

三维非齐次磁流体动力学方程在有界密度和真空情形下的全局稳定性

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摘要: 本文研究具有有界密度的三维 (3D) 非齐次磁流体动力学 (MHD) 系统的全局稳定性。应用拉格朗日坐标方法, 我们证明了 Liu 和 Liu (Commun. Math. Sci. 20 (2022), pp. 1517–1540) 所获得的全局强解的指数稳定性。特别地, 尽管存在真空, 也无需对初始数据施加某些相容性条件。

关键词: 非齐次磁流体方程; 拉格朗日坐标; 全局稳定性; 有界密度; 真空。

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Global stability for the 3D inhomogeneous magnetohydrodynamic equations with bounded density and vacuum

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Abstract: This paper is concerned with the global stability for three-dimensional (3D) inhomogeneous magnetohydrodynamic (MHD) system with bounded density. Applying the Lagrangian coordinate method, we prove the exponential stability of global strong solutions obtained by Liu and Liu (Commun. Math. Sci. 20 (2022), pp. 1517–1540). In particular, there is no need to impose some compatibility condition on the initial data despite the presence of vacuum.

Key words: Inhomogeneous MHD equations; Lagrangian coordinates; global stability; bounded density; vacuum.

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0 Introduction

0.1 Background and motivation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, we investigate the inhomogeneous MHD equations in $\Omega \times (0, T)$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) - \mu \Delta \mathbf{v} + \nabla P = \nabla \times \mathbf{H} \times \mathbf{H}, \\ \mathbf{H}_t + \nu \nabla \times \nabla \times \mathbf{H} = \nabla \times (\mathbf{v} \times \mathbf{H}), \\ \operatorname{div} \mathbf{H} = 0, \operatorname{div} \mathbf{v} = 0. \end{cases} \quad (1)$$

Here ρ , $\mathbf{v}$, P , and $\mathbf{H}$ represent the fluid density, velocity field, pressure, and magnetic field, respectively. $\mu > 0$ is the viscosity coefficient, while $\nu > 0$ stands for the magnetic diffusion coefficient.

We consider an initial-boundary value problem of system (1) with the initial condition

$$(\rho, \mathbf{v}, \mathbf{H})(x, 0) = (\rho_0, \mathbf{v}_0, \mathbf{H}_0), \quad x \in \Omega, \quad (2)$$

and the boundary conditions

$$\mathbf{v} = 0, \quad \mathbf{H} \cdot \mathbf{n} = 0, \quad \operatorname{curl} \mathbf{H} \times \mathbf{n} = \mathbf{0}, \quad x \in \partial\Omega, \quad t > 0, \quad (3)$$

where $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$.

The inhomogeneous MHD equations govern the motion of conducting fluids interacting with magnetic fields. The fluid density is no longer constant but satisfies a transport equation, while the velocity field and the magnetic field are mutually coupled through the Lorentz force and the magnetic induction equation. In recent years, substantial progress has been made on the global well-posedness for such a model. It was showed in [1] that the Cauchy problem of (1) has a unique global weak solution if the initial data are sufficiently small in critical Besov spaces. Later on, without any small conditions imposed on the third component of the initial velocity field and magnetic field in the critical Besov spaces, the global well-posedness of strong solutions was obtained in [2] with the aid of the Fourier frequency localization and Bony's paraproduct decomposition; see related work [3]. Under the condition that $(\rho_0, \mathbf{v}_0, \mathbf{H}_0) \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3) \times \dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3) \times \dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3)$ with $1 \leq q \leq p$, $p \in (1, (5 + \sqrt{17})/2)$, and $\frac{1}{2} - \frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{p} + \frac{1}{3}$, Zhai and Yin [4] derived a global unique solutions when the initial velocity is small enough in $\dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3)$. Moreover, based on the Lagrangian approach, Bie-Wang-Yao [5] established a unique global solutions provided that the initial data $(\rho_0 - 1, \mathbf{v}_0, \mathbf{H}_0) \in \mathcal{M}(\dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3)) \times \dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3) \times \dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3)$ satisfies some smallness condition, where $1 \leq p < 6$ and $\mathcal{M}(\dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3))$ is the multiplier space of $\dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3)$. Huang and Qian [6] extended ρ_0 to $\dot{B}_q^{3/q}$ and took $(\mathbf{v}_0, \mathbf{H}_0)$

in $\dot{B}_p^{-1+3/p}$, where $1 < p, q < 6$ satisfy $\max \left\{ \frac{1}{2} - \frac{1}{q}, \frac{1}{q} - \frac{1}{3} \right\} < \frac{1}{p} < \frac{1}{6} + \frac{1}{q}$, thereby removing the previous restriction $1 < q \leq p$, and established global existence and uniqueness under suitable smallness conditions. Recently, Chen–Guo–Zhai [7] proved the global well-posedness and decay estimate of solutions when $(\rho_0, \mathbf{v}_0, \mathbf{H}_0) \in L^\infty(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ with $\frac{1}{2} < s \leq 1$ and $\|(\mathbf{v}_0, \mathbf{H}_0)\|_{\dot{H}^{\frac{1}{2}}}$ is sufficiently small.

It should be pointed out that all the results stated above are devoted to the non-vacuum case. Once the initial density vanishes on some domain, the analysis becomes more delicate because the system degenerates in the vacuum region. Nevertheless, there are also many works regarding this interesting setting. Under a natural compatibility condition

$$-\mu \Delta \mathbf{v}_0 + \nabla P_0 - (\mathbf{H}_0 \cdot \nabla) \mathbf{H}_0 = \sqrt{\rho_0} g \quad \text{for some } (P_0, g) \in H^1 \times L^2, \quad (4)$$

the global existence of strong solutions to the 3D Cauchy problem of (1) was obtained in [8] under the smallness assumption of initial data, while Huang and Wang [9] showed the global existence of strong solutions to the 2D Dirichlet problem involving large initial data with the aid of Brezis–Wainger type Sobolev inequality of logarithmic type. Subsequently, the second author [10] removed the compatibility condition (4) by time-weighted techniques. Meanwhile, in another paper [11] in collaboration with Lü and Xu, the second author tackled the 2D Cauchy problem provided that the initial density and the initial magnetic field decay not too slow at infinity. It is worth emphasizing that the initial density in [8–11] is imposed to Sobolev spaces in order to ensure the uniqueness of solutions. Recently, by virtue of Lagrangian coordinates method introduced in [12], it was proved in [13] that the system (1) has a global unique strong solution in 3D tori or bounded domains when the quantity

$$(\|\sqrt{\rho_0} \mathbf{v}_0\|_{L^2}^2 + \|\mathbf{H}_0\|_{L^2}^2) (\|\nabla \mathbf{v}_0\|_{L^2}^2 + \|\nabla \mathbf{H}_0\|_{L^2}^2 + \|\mathbf{H}_0\|_{L^4}^4)$$

is properly small. At the same time, Liu and Liu [14] demonstrated the uniqueness of global strong solutions of (1) in 3D bounded domains as long as

$$(\|\sqrt{\rho_0} \mathbf{v}_0\|_{L^2}^2 + \|\mathbf{H}_0\|_{L^2}^2) (\|\nabla \mathbf{v}_0\|_{L^2}^2 + \|\nabla \mathbf{H}_0\|_{L^2}^2) \leq \varepsilon \quad (5)$$

for some small constant ε .

Some important progress has also been made concerning the global stability and long-time behavior of solutions to the inhomogeneous MHD equations. In the absence of vacuum, the large-time decay and stability to any given global solutions of the 3D incompressible MHD system in Besov spaces were investigated in [15, 16], while Chen–Guo–Zhai [7] obtained the decay estimate of global solutions in the framework of Sobolev spaces. When interior vacuum is allowed in a bounded domain $\Omega \subset \mathbb{R}^3$, Liu and Liu [14] proved

$$\|\mathbf{v}(t)\|_{W^{1,p}}^2 + \|\mathbf{H}(t)\|_{W^{1,p}}^2 \leq C e^{-\sigma t}, \quad \text{for all } t > 1, p \in [2, 6). \quad (6)$$

However, the global stability of strong solutions with bounded density derived in [14] is still unknown. In fact, this is the primary objective of this paper.

0.2 Main result

Before stating our main results, we first introduce the notation used throughout the paper. For simplicity, we write

$$\int f dx \triangleq \int_{\Omega} f(x) dx.$$

For $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, we use $L^r = L^r(\Omega)$ and $W^{k,r} = W^{k,r}(\Omega)$ to stand for the standard Lebesgue and Sobolev spaces, respectively. When $r = 2$, we use $H^k \triangleq W^{k,2}(\Omega)$. Define

$$H_0^1 \triangleq \{\mathbf{u} \in H^1 \mid \mathbf{u} = 0 \text{ on } \partial\Omega\}, \quad H_n^1 \triangleq \{\mathbf{u} \in H^1 \mid \mathbf{u} \cdot \mathbf{n} = 0, \quad (\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{0}, \text{ on } \partial\Omega\}.$$

We denote by $W^{-1,p}(\Omega) \triangleq (W_0^{1,p'}(\Omega))^*$, where $\frac{1}{p} + \frac{1}{p'} = 1$. For an interval $I \subset \mathbb{R}$ and a Banach space X , $L^p(I; X)$ denotes the usual Bochner space. Throughout the paper, C and C_p ($2 \leq p \leq 6$) stand for generic positive constants which may depend on the upper bound ρ^* of the initial density, Ω , μ , ν , and the initial data. We also use $f_p(t)$ to denote generic functions in $L^p(\mathbb{R}_+)$, whose explicit expressions may vary from line to line. Furthermore, we consider two solutions $(\rho^i, \mathbf{v}^i, \mathbf{H}^i)$, $i = 1, 2$, corresponding to the initial data $(\rho_0^i, \mathbf{v}_0^i, \mathbf{H}_0^i)$. Their Lagrangian representations are denoted by $(\eta^i, \mathbf{u}^i, \mathbf{d}^i, Q^i)$ and will be defined precisely in Section 2.

We recall the global well-posedness result for system (1) established in [14].

Proposition 1. *Let ρ^* be a positive constant and assume that the initial data $(\rho_0, \mathbf{v}_0, \mathbf{H}_0)$ satisfies*

$$0 \leq \rho_0 \leq \rho^*, \quad (\mathbf{v}_0, \mathbf{H}_0) \in H_0^1 \times H_n^1, \quad \operatorname{div} \mathbf{v}_0 = 0, \quad \operatorname{div} \mathbf{H}_0 = 0 \quad (7)$$

and small condition (5), then the initial-boundary value problem (1)–(3) admits a unique global solution satisfying, for any $0 < \tau < +\infty$,

$$\left\{ \begin{array}{l} 0 \leq \rho \in L^\infty(0, +\infty; L^\infty) \cap C([0, +\infty); L^q), \quad \sqrt{\rho} \mathbf{v} \in C([0, +\infty); L^2), \\ \mathbf{v} \in L^\infty(0, +\infty; H_0^1) \cap L^2(0, +\infty; H^2) \cap L^2(\tau, +\infty; W^{2,6}) \cap C([\tau, +\infty); W^{1,p}), \\ \nabla P \in L^2(\tau, +\infty; L^6) \cap L^2(0, +\infty; L^2), \\ \mathbf{H} \in L^\infty(0, +\infty; H_n^1) \cap L^2(0, +\infty; H^2) \cap L^2(\tau, +\infty; W^{2,6}) \cap C([\tau, +\infty); W^{1,p}), \\ \sqrt{\rho} \mathbf{v}_t \in L^2(0, +\infty; L^2) \cap L^\infty(\tau, +\infty; L^2), \quad \mathbf{v}_t \in L^2(\tau, +\infty; H_0^1), \\ \mathbf{H}_t \in L^2(0, +\infty; L^2) \cap L^\infty(\tau, +\infty; L^2) \cap L^2(\tau, +\infty; H_n^1), \end{array} \right. \quad (8)$$

where $q \in [1, +\infty)$ and $p \in [2, 6)$.

Now we can state the main results of this paper.

Theorem 1. *Assume that the initial data $(\rho_0^i \geq 0, \mathbf{v}_0^i, \mathbf{H}_0^i)$ ($i = 1, 2$) satisfy (5) and (7), and $(\rho^i \geq 0, \mathbf{v}^i, \mathbf{H}^i)$ are the corresponding global strong solutions given by Proposition 1. Denote by*

$$\delta \mathbf{u} \triangleq \mathbf{u}^2 - \mathbf{u}^1, \quad \delta \mathbf{v}_0 \triangleq \mathbf{v}_0^2 - \mathbf{v}_0^1, \quad \delta \rho \triangleq \rho^2 - \rho^1, \quad \delta \mathbf{H}_0 \triangleq \mathbf{H}_0^2 - \mathbf{H}_0^1, \quad \delta \mathbf{d} \triangleq \mathbf{d}^2 - \mathbf{d}^1.$$

There exists a positive constant β depending only on μ, ν, ρ^* , and Ω such that, for any $t \geq 0$,

$$\begin{aligned} & e^{\beta t} \left(\left\| \min \{(\rho_0^1)^{1/2}, (\rho_0^2)^{1/2}\} \delta \mathbf{u} \right\|_{L^2} + \|\delta \mathbf{d}\|_{L^2} \right) + \mu^{\frac{1}{2}} \|e^{\beta t} \nabla \delta \mathbf{u}\|_{L^2(0,t;L^2)} + \nu^{\frac{1}{2}} \|e^{\beta t} \nabla \delta \mathbf{d}\|_{L^2(0,t;L^2)} \\ & \leq C \left(\|(\rho_0^1)^{1/2} \delta \mathbf{v}_0\|_{L^2} + \|\delta \mathbf{H}_0\|_{L^2} + \|\delta \rho_0\|_{L^2}^{\frac{1}{2}} \right), \end{aligned} \quad (9)$$

where $\rho^* \triangleq \max\{\|\rho_0^1\|_{L^\infty}, \|\rho_0^2\|_{L^\infty}\}$ and $C = C(\Omega, \mu, \nu, \rho_0^1, \rho_0^2, \mathbf{v}_0^1, \mathbf{v}_0^2, \mathbf{H}_0^1, \mathbf{H}_0^2)$.

Next, coming back from Lagrangian to Eulerian coordinates, we obtain the following stability result.

Theorem 2. *Under the assumptions in Theorem 1, we have*

$$\begin{aligned} & \sup_{t \in \mathbb{R}_+} \|\delta \rho(t)\|_{W^{-1,p}} + \sup_{t \in \mathbb{R}_+} \left[e^{\beta t} \left(\|\sqrt{\rho^1} \delta \mathbf{v}(t)\|_{L^2} + \|\delta \mathbf{H}(t)\|_{L^2} \right) \right] \\ & + \mu^{\frac{1}{2}} \|e^{\beta t} \nabla \delta \mathbf{v}\|_{L^2(\mathbb{R}_+;L^2)} + \nu^{\frac{1}{2}} \|e^{\beta t} \nabla \delta \mathbf{H}\|_{L^2(\mathbb{R}_+;L^2)} \\ & \leq C \left(\|(\rho_0^1)^{1/2} \delta \mathbf{v}_0\|_{L^2} + \|\delta \mathbf{H}_0\|_{L^2} + \|\delta \rho_0\|_{L^2}^{\frac{1}{2}} \right), \end{aligned} \quad (10)$$

where $1 < p \leq 6$, $\delta \mathbf{v} \triangleq \mathbf{v}^2 - \mathbf{v}^1$, and $\delta \mathbf{H} \triangleq \mathbf{H}^2 - \mathbf{H}^1$.

Remark 1. *It should be noted that the global existence of a unique strong solution to the problem (1)–(3) was obtained in [14], while we provide the stability counterpart for such solutions, thereby complementing the global well-posedness theory. To the best of our knowledge, Theorems 1 and 2 are the first results on the stability of the inhomogeneous MHD equations with bounded density and vacuum.*

Remark 2. *We emphasize that the boundedness of domains plays an crucial role in our analysis, and it is still not clear how to show the stability of (1) with bounded density and vacuum in unbounded domains.*

Remark 3. *It is worth mentioning that the uniqueness of a global strong solution of (1) with large initial data and vacuum in 2D bounded domains or $\mathbb{R}^2$ was established in [9, 11] when the density has enough regularity, guaranteeing the Lipschitz control of the flow of the velocity field. Naturally, one may wonder whether we can establish the global well-posedness theory for the system (1) in 2D domains with only bounded density and vacuum. We leave this interesting problem to the future research.*

We now outline the main strategies and challenges. The main difficulties come from the low regularity of the density and the possible presence of vacuum. On the one hand, in the vacuum region, the density-weighted kinetic energy does not provide coercive control of the velocity perturbation. On the other hand, since the density is merely bounded, a direct stability analysis in Eulerian coordinates may not be available. Indeed, taking the difference of $(1)_1$ satisfied by ρ^1 and ρ^2 gives

$$\delta \rho_t + \mathbf{v}^1 \cdot \nabla \delta \rho = -\delta \mathbf{v} \cdot \nabla \rho^2,$$

where the nonlinear transport caused by $\delta \mathbf{v} = \mathbf{v}^2 - \mathbf{v}^1$ is not spatially integrable when $\rho^2 \in L^\infty$. Motivated by [17], where Danchin, Mucha, and Piasecki investigated the stability for the inhomogeneous incompressible fluids, we work in Lagrangian coordinates, where the density remains constant along particle trajectories. We point out that the Lagrangian stability argument in [17] seems not to be applied here directly since the additional nonlinear couplings $\mathbf{v} \cdot \nabla \mathbf{H}$ and $\mathbf{H} \cdot \nabla \mathbf{v}$ bring new difficulties. Some new estimates need to be explored, especially in the presence of the physical boundary of the magnetic field.

Before the stability analysis, we first establish the global-in-time decay and the time-weighted estimates. Starting from the basic energy identity, the $L^\infty(\mathbb{R}^+; L^2(\Omega))$ -norm of $\sqrt{\rho} \mathbf{v}$ and $\mathbf{H}$ are obtained (see Lemma 2). Next, testing (1)₂ and (1)₃ against $\mathbf{v}_t$ and $\mathbf{H}_t$, respectively, together with the exponential decay (6) obtained in [14], the exponential decay of $\nabla \mathbf{v}$ and $\nabla \mathbf{H}$ is established in Lemma 3. To better study the momentum equation (1)₂, we introduce the time weight $\sqrt{t}e^{\beta t}$ and derive weighted energy estimates for $\mathbf{v}_t$ and $\mathbf{H}_t$ (see Lemma 4). Combining these estimates, the bounds of $\sqrt{t}e^{\beta t}(\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P)$ are showed (see (40)). To deal with the singularity of t at $t = 0$, inspired by [17], we split the time interval into $(0, 1)$ and $(1, \infty)$, and obtain the $L^1(\mathbb{R}_+; L^r)$ -norm of $e^{\beta t}(\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H})$ (see (41)). Furthermore, the $L^1(\mathbb{R}_+; L^\infty)$ -norm of $e^{\beta t}(\nabla \mathbf{v}, \nabla \mathbf{H})$ is established in (42). These estimates play essential roles in the subsequent stability analysis within the Lagrangian framework.

Next, we turn to the stability argument. It should be noted that the Lagrangian method bypasses the L^p estimates of $\nabla \rho$. Nevertheless, the change of coordinates still gives rise to new difficulties associated with the emergence of the Jacobian matrix. Along the particle trajectories of the two solutions, we have to consider the corresponding flow maps $X^{1,2}$ and deformation matrices $A^{1,2}$. Indeed, $\delta \mathbf{u} = \mathbf{u}^2 - \mathbf{u}^1$ fails to inherit the divergence-free condition for the solutions $\mathbf{u}^1$ and $\mathbf{u}^2$. To recover a suitable divergence-free structure, we decompose $\delta \mathbf{u} = \widehat{\mathbf{z}} + \widetilde{\mathbf{z}}$, where $\widetilde{\mathbf{z}}$ satisfies $\text{div}_{\mathbf{u}^1} \widetilde{\mathbf{z}} = 0$ and $\widehat{\mathbf{z}}$ is introduced to correct the divergence defect (see (56)). For the magnetic field $\delta \mathbf{d} = \mathbf{d}^2 - \mathbf{d}^1$, based on the slip boundary condition and using the divergence-free properties of $\mathbf{d}^1$ and $\mathbf{d}^2$, we test $\Delta \delta \mathbf{d}$ and estimate $\|\text{curl}_{\mathbf{u}^1} \delta \mathbf{d}\|_{L^2}$, whereas $|\text{div}_{\mathbf{u}^1} - \text{div}_{\mathbf{u}^2}|$ is controlled by the difference of the deformation matrices. Having these estimates at hand, together with the bounds of $\delta A = A^2 - A^1$ (see (57)) and $\widehat{\mathbf{z}}$ (see (58)–(60)), the exponential decay of $\delta \mathbf{u}$ and $\delta \mathbf{d}$ (see (86)) can be established by Gronwall's inequality in the case of the non-vacuum setting. However, the vacuum case requires additional observation. To overcome it, we introduce the intermediate initial density $\rho_0^{3/2} = \frac{1}{2}(\rho_0^1 + \rho_0^2)$ and consider the corresponding intermediate solution $(\eta^{3/2}, \mathbf{u}^{3/2}, \mathbf{d}^{3/2}, Q^{3/2})$ in Lagrangian coordinates. The original perturbation $\delta \mathbf{u}$ and $\delta \mathbf{d}$ are then decomposed into $\delta \mathbf{u}^{\text{I,II}}$ and $\delta \mathbf{d}^{\text{I,II}}$, respectively. With the aid of this construction, we can obtain an upper bound on the density

difference, namely,

$$\left\| \frac{\delta \rho_0^I}{\sqrt{\rho_0^1 + \rho_0^2}} \right\|_{L^4} + \left\| \frac{\delta \rho_0^H}{\sqrt{\rho_0^1 + \rho_0^2}} \right\|_{L^4} \leq C \|\rho_0^1 - \rho_0^2\|_{L^2}^{1/2};$$

see (88) for details. The resulting estimates complete the proof of the global stability in Theorem 1.

Finally, we return from the Lagrangian coordinates to the original Eulerian coordinates. $\delta \mathbf{v}$ and $\delta \mathbf{H}$ are controlled by means of the estimates for the corresponding flow maps, while $\delta \rho$ is measured in a negative Sobolev space, which avoids the requirement of any spatial derivative of the density. In this way, we obtain the global-in-time stability estimates for the density, velocity, and magnetic field in Eulerian coordinates, shown in Theorem 2.

The remainder of this paper is organized as follows. Section 1 is devoted to establishing the exponential decay estimates of the solution. Then we give the proof of Theorem 1 in Section 2, while the proof of Theorem 2 is presented in Section 3.

1 Exponential decay estimate

In this section we establish the exponential decay estimates for solutions. First of all, it follows from (1)₁ and (1)₄ that the density satisfies $\|\rho(t)\|_{L^p} = \|\rho_0\|_{L^p}$ for any $p \in [1, \infty]$. Moreover, for any $\mathbf{v} \in H^1(\Omega)$, there is a $C > 0$ depending on Ω such that

$$\|\mathbf{v}\|_{L^2} \leq C \|\nabla \mathbf{v}\|_{L^2} \quad (11)$$

provided $\int \mathbf{v} dx = 0$ or $(\mathbf{v} \cdot \mathbf{n})|_{\partial\Omega} = 0$ or $(\mathbf{v} \times \mathbf{n})|_{\partial\Omega} = \mathbf{0}$ (see [18, Lemma 8]). Furthermore, the following Gagliardo–Nirenberg inequality (see, e.g., [19, Lemma 2.3]) will be used frequently.

Lemma 1 (Gagliardo–Nirenberg inequalities: special cases). *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. Denote by $\mathbf{n}$ the unit outward normal on $\partial\Omega$. For any $f \in H^1(\Omega)$ and $g \in L^q(\Omega)$ with $\nabla g \in L^s(\Omega)$, there exist constants $C_i \geq 0$ ($i = 1, 2, 3, 4$) depending only on p, q, s , and Ω such that*

$$\begin{aligned} \|f\|_{L^p(\Omega)} &\leq C_1 \|f\|_{L^2(\Omega)}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2(\Omega)}^{\frac{3p-6}{2p}} + C_2 \|f\|_{L^2(\Omega)}, \\ \|g\|_{L^\infty(\Omega)} &\leq C_3 \|g\|_{L^q(\Omega)}^{\frac{q(s-3)}{3s+q(s-3)}} \|\nabla g\|_{L^s(\Omega)}^{\frac{3s}{3s+q(s-3)}} + C_4 \|g\|_{L^2(\Omega)}, \end{aligned}$$

where $p \in [2, 6]$, $q \in (1, \infty)$, $s \in (3, \infty)$. Furthermore, if $\int_\Omega f dx = 0$ or $(f \cdot \mathbf{n})|_{\partial\Omega} = 0$, we can choose $C_2 = 0$. Similarly, we can choose $C_4 = 0$ provided $(g \cdot \mathbf{n})|_{\partial\Omega} = 0$.

We begin with the following basic energy estimate for a global strong solution $(\rho, \mathbf{v}, \mathbf{H})$, which is given by Proposition 1.

Lemma 2. *There exists a positive constant β_1 depending only on μ, ν, ρ^* , and Ω such that, for all $t \geq 0$,*

$$\|\sqrt{\rho}\mathbf{v}\|_{L^2}^2 + \|\mathbf{H}\|_{L^2}^2 \leq e^{-2\beta_1 t} (\|\sqrt{\rho_0}\mathbf{v}_0\|_{L^2}^2 + \|\mathbf{H}_0\|_{L^2}^2). \quad (12)$$

Proof. Multiplying (1)₂ and (1)₃ by $\mathbf{v}$ and $\mathbf{H}$, respectively, summing them, and integrating the resultant over Ω , we obtain that

$$\frac{1}{2} \frac{d}{dt} \int (\rho|\mathbf{v}|^2 + |\mathbf{H}|^2) dx + \mu \int |\nabla \mathbf{v}|^2 dx + \nu \int |\nabla \times \mathbf{H}|^2 dx = 0. \quad (13)$$

Set $\sigma^1 \triangleq \min\{\mu, \nu\}$. Using (11) and the inequality $\|\nabla \mathbf{H}\|_{L^2} \leq C\|\nabla \times \mathbf{H}\|_{L^2}$, we have

$$\sigma^1 (\|\mathbf{v}\|_{L^2}^2 + \|\mathbf{H}\|_{L^2}^2) \leq C(\mu\|\nabla \mathbf{v}\|_{L^2}^2 + \nu\|\nabla \times \mathbf{H}\|_{L^2}^2),$$

which together with (13) gives

$$\frac{d}{dt} (\|\sqrt{\rho}\mathbf{v}\|_{L^2}^2 + \|\mathbf{H}\|_{L^2}^2) + 2\frac{\sigma^1}{C} (\|\mathbf{v}\|_{L^2}^2 + \|\mathbf{H}\|_{L^2}^2) \leq 0. \quad (14)$$

Recall the uniform upper bound $0 \leq \rho \leq \rho^*$, which implies $\|\sqrt{\rho}\mathbf{v}\|_{L^2}^2 \leq \rho^*\|\mathbf{v}\|_{L^2}^2$. Set $\sigma^2 \triangleq \min\{1, \rho^*\}$, $\sigma^* \triangleq \sigma^1 \sigma^2$, and $\beta_1 \triangleq \frac{\sigma^*}{C\rho^*}$. Multiplying (14) by ρ^* and integrating the resultant with respect to time over $(0, t)$ leads to the desired (12). $\square$

Next, we establish a key uniform-in-time estimate for the $L^\infty(\mathbb{R}^+; L^2)$ -norm of the gradients of the velocity and the magnetic field.

Lemma 3. *There exists a positive constant $\beta_2 \leq \beta_1$ depending only on μ, ν, ρ^* , and Ω such that, for all $t \geq 0$,*

$$\int_0^\infty e^{2\beta_2 t} (\|\sqrt{\rho}\mathbf{v}_t\|_{L^2}^2 + \|\mathbf{H}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{v}\|_{L^2}^2 + \|\nabla^2 \mathbf{H}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2) dt \leq C, \quad (15)$$

$$\|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2 \leq C e^{-2\beta_2 t}, \quad (16)$$

$$\|(\mathbf{v}, \mathbf{H})\|_{L^p}^2 \leq C_p e^{-2\beta_2 t} \quad \text{for any } p \in (2, 6]. \quad (17)$$

Proof. Multiplying (1)₂ and (1)₃ by $\mathbf{v}_t$ and $\mathbf{H}_t$, respectively, summing the resulting identities, and integrating over Ω , we obtain

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \|\nabla \mathbf{v}\|_{L^2}^2 + \frac{\nu}{2} \frac{d}{dt} \|\nabla \times \mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho}\mathbf{v}_t\|_{L^2}^2 + \|\mathbf{H}_t\|_{L^2}^2 \\ & \leq \frac{d}{dt} \int (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} dx + \int (\mathbf{H}_t \cdot \nabla) \mathbf{v} \cdot \mathbf{H} dx + 2 \int (\mathbf{H} \cdot \nabla) \mathbf{v} \cdot \mathbf{H}_t dx \\ & \quad + \frac{1}{2} \int \rho |\mathbf{v} \cdot \nabla \mathbf{v}|^2 dx + \int |(\mathbf{v} \cdot \nabla) \mathbf{H} \cdot \mathbf{H}_t| dx. \end{aligned} \quad (18)$$

To derive the second-order spatial estimates for $\mathbf{v}$ and $\mathbf{H}$, we apply the standard L^2 -regularity estimate for the Stokes system and the elliptic regularity estimate to (1)₂ and (1)₃,

respectively, obtaining

$$\begin{cases} \|\nabla^2 \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \leq C_\Omega (\|\rho \mathbf{v}_t\|_{L^2}^2 + \|\rho \mathbf{v} \cdot \nabla \mathbf{v}\|_{L^2}^2 + \|\mathbf{H} \cdot \nabla \mathbf{H}\|_{L^2}^2), \\ \|\nabla^2 \mathbf{H}\|_{L^2}^2 \leq C_\Omega (\|\mathbf{H}_t\|_{L^2}^2 + \|(\mathbf{H} \cdot \nabla) \mathbf{v}\|_{L^2}^2 + \|(\mathbf{v} \cdot \nabla) \mathbf{H}\|_{L^2}^2). \end{cases} \quad (19)$$

Then, multiplying (19)₁ by 1/2 and (19)₂ by 3/2, respectively, and adding the resulting estimates to (18), we arrive at

$$\begin{aligned} & \frac{d}{dt} (\mu \|\nabla \mathbf{v}\|_{L^2}^2 + \nu \|\nabla \times \mathbf{H}\|_{L^2}^2) + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}_t\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}_t\|_{L^2}^2 + \frac{1}{2C_\Omega} (\|\nabla^2 \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 \mathbf{H}\|_{L^2}^2) \\ & \leq 2 \frac{d}{dt} \int (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} \, dx + 2 \int (\mathbf{H}_t \cdot \nabla) \mathbf{v} \cdot \mathbf{H} \, dx + \frac{3}{2} \int \rho |\mathbf{v} \cdot \nabla \mathbf{v}|^2 \, dx + 4 \int |(\mathbf{H} \cdot \nabla) \mathbf{v} \cdot \mathbf{H}_t| \, dx \\ & \quad + 2 \int |(\mathbf{v} \cdot \nabla) \mathbf{H} \cdot \mathbf{H}_t| \, dx + \frac{1}{2} \|\mathbf{H} \cdot \nabla \mathbf{H}\|_{L^2}^2 + \frac{3}{2} (\|(\mathbf{H} \cdot \nabla) \mathbf{v}\|_{L^2}^2 + \|(\mathbf{v} \cdot \nabla) \mathbf{H}\|_{L^2}^2). \end{aligned} \quad (20)$$

Hence, inserting (13) into (20) leads to

$$\begin{aligned} & \frac{d}{dt} \left(\mu \|\nabla \mathbf{v}\|_{L^2}^2 + \nu \|\nabla \times \mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}\|_{L^2}^2 \right) + \mu \|\nabla \mathbf{v}\|_{L^2}^2 + \nu \|\nabla \times \mathbf{H}\|_{L^2}^2 \\ & + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}_t\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}_t\|_{L^2}^2 + \frac{1}{2C_\Omega} (\|\nabla^2 \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 \mathbf{H}\|_{L^2}^2) \\ & \leq 2 \frac{d}{dt} \int (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} \, dx + 2 \int (\mathbf{H}_t \cdot \nabla) \mathbf{v} \cdot \mathbf{H} \, dx + \frac{3}{2} \int \rho |\mathbf{v} \cdot \nabla \mathbf{v}|^2 \, dx + 4 \int |(\mathbf{H} \cdot \nabla) \mathbf{v} \cdot \mathbf{H}_t| \, dx \\ & \quad + 2 \int |(\mathbf{v} \cdot \nabla) \mathbf{H} \cdot \mathbf{H}_t| \, dx + \frac{1}{2} \|\mathbf{H} \cdot \nabla \mathbf{H}\|_{L^2}^2 + \frac{3}{2} (\|(\mathbf{H} \cdot \nabla) \mathbf{v}\|_{L^2}^2 + \|(\mathbf{v} \cdot \nabla) \mathbf{H}\|_{L^2}^2) \\ & \quad + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}\|_{L^2}^2. \end{aligned} \quad (21)$$

By virtue of (11), Sobolev's, Hölder's, and Gagliardo–Nirenberg inequalities, we find that

$$\begin{aligned} 2 \int (\mathbf{H}_t \cdot \nabla) \mathbf{v} \cdot \mathbf{H} \, dx + 4 \int |(\mathbf{H} \cdot \nabla) \mathbf{v} \cdot \mathbf{H}_t| \, dx & \leq C \|\mathbf{H}_t\|_{L^2} \|\mathbf{H}\|_{L^6} \|\nabla \mathbf{v}\|_{L^3} \\ & \leq C \|\mathbf{H}_t\|_{L^2} \|\nabla \mathbf{H}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \mathbf{v}\|_{L^2}^{\frac{1}{2}} \\ & \leq \frac{1}{8} \|\mathbf{H}_t\|_{L^2}^2 + \frac{1}{12C_\Omega} \|\nabla^2 \mathbf{v}\|_{L^2}^2 + C \|\nabla \mathbf{H}\|_{L^2}^4 \|\nabla \mathbf{v}\|_{L^2}^2, \\ 2 \int |(\mathbf{v} \cdot \nabla) \mathbf{H} \cdot \mathbf{H}_t| \, dx & \leq C \|\mathbf{H}_t\|_{L^2} \|\nabla \mathbf{H}\|_{L^3} \|\mathbf{v}\|_{L^6} \\ & \leq CC_1 \|\mathbf{H}_t\|_{L^2} \|\nabla \mathbf{v}\|_{L^2} \|\nabla \mathbf{H}\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \mathbf{H}\|_{L^2}^{\frac{1}{2}} \\ & \quad + CC_2 \|\mathbf{H}_t\|_{L^2} \|\nabla \mathbf{v}\|_{L^2} \|\nabla \mathbf{H}\|_{L^2} \\ & \leq \frac{1}{8} \|\mathbf{H}_t\|_{L^2}^2 + \frac{1}{12C_\Omega} \|\nabla^2 \mathbf{H}\|_{L^2}^2 + C \|\nabla \mathbf{H}\|_{L^2}^2 \|\nabla \mathbf{v}\|_{L^2}^2 \\ & \quad + C \|\nabla \mathbf{H}\|_{L^2}^2 \|\nabla \mathbf{v}\|_{L^2}^4, \\ \frac{3}{2} \int |(\mathbf{H} \cdot \nabla) \mathbf{v}|^2 \, dx + \frac{3}{2} \int |(\mathbf{v} \cdot \nabla) \mathbf{H}|^2 \, dx & \leq C \|\mathbf{H}\|_{L^6}^2 \|\nabla \mathbf{v}\|_{L^3}^2 + C \|\mathbf{v}\|_{L^6}^2 \|\nabla \mathbf{H}\|_{L^3}^2 \\ & \leq \frac{1}{12C_\Omega} \|\nabla^2 \mathbf{v}\|_{L^2}^2 + \frac{1}{12C_\Omega} \|\nabla^2 \mathbf{H}\|_{L^2}^2 \\ & \quad + C \|\nabla \mathbf{v}\|_{L^2}^2 \|\nabla \mathbf{H}\|_{L^2}^2 (1 + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2). \end{aligned}$$

Similarly, we derive from (12) that

$$\begin{aligned}
 \frac{3}{2} \int \rho |\mathbf{v} \cdot \nabla \mathbf{v}|^2 dx &\leq \frac{3}{2} \sqrt{\rho^*} \|\rho^{1/4} \mathbf{v}\|_{L^4}^2 \|\nabla \mathbf{v}\|_{L^4}^2 \leq C \|\rho^{1/4} \mathbf{v}\|_{L^2}^{\frac{1}{2}} \|\rho^{1/2} \mathbf{v}\|_{L^6}^{\frac{3}{2}} \|\nabla \mathbf{v}\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \mathbf{v}\|_{L^2}^{\frac{3}{2}} \\
 &\leq C \|\rho^{1/2} \mathbf{v}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2}^2 \|\nabla^2 \mathbf{v}\|_{L^2}^{\frac{3}{2}} \\
 &\leq \frac{1}{12C_\Omega} \|\nabla^2 \mathbf{v}\|_{L^2}^2 + C \|\sqrt{\rho} \mathbf{v}\|_{L^2}^2 \|\nabla \mathbf{v}\|_{L^2}^8 \\
 &\leq \frac{1}{12C_\Omega} \|\nabla^2 \mathbf{v}\|_{L^2}^2 + C e^{-2\beta_1 t} \|\nabla \mathbf{v}\|_{L^2}^8, \\
 \frac{1}{2} \int |\mathbf{H} \cdot \nabla \mathbf{H}|^2 dx &\leq C \|\mathbf{H}\|_{L^4}^2 \|\nabla \mathbf{H}\|_{L^4}^2 \leq C \|\mathbf{H}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{H}\|_{L^2}^{\frac{3}{2}} C_1 \|\nabla \mathbf{H}\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \mathbf{H}\|_{L^2}^{\frac{3}{2}} \\
 &\quad + C C_2 \|\mathbf{H}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{H}\|_{L^2}^{\frac{3}{2}} \|\nabla \mathbf{H}\|_{L^2}^2 \\
 &\leq C \|\mathbf{H}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{H}\|_{L^2}^2 \|\nabla^2 \mathbf{H}\|_{L^2}^{\frac{3}{2}} + C \|\nabla \mathbf{H}\|_{L^2}^4 \\
 &\leq \frac{1}{12C_\Omega} \|\nabla^2 \mathbf{H}\|_{L^2}^2 + C \|\mathbf{H}\|_{L^2}^2 \|\nabla \mathbf{H}\|_{L^2}^8 + C \|\nabla \mathbf{H}\|_{L^2}^4 \\
 &\leq \frac{1}{12C_\Omega} \|\nabla^2 \mathbf{H}\|_{L^2}^2 + C e^{-2\beta_1 t} \|\nabla \mathbf{H}\|_{L^2}^8 + C \|\nabla \mathbf{H}\|_{L^2}^4.
 \end{aligned}$$

Setting $c^* \triangleq \min\{1, \mu, \nu\}$, and substituting the above estimates into (21), we obtain from Lemma 2 that

$$\begin{aligned}
 &\frac{d}{dt} \left(\|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \times \mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}\|_{L^2}^2 \right) + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \times \mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}\|_{L^2}^2 \\
 &+ \frac{1}{2} \|\mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}_t\|_{L^2}^2 + \frac{1}{4} \|\mathbf{H}_t\|_{L^2}^2 + \frac{1}{4C_\Omega} (\|\nabla^2 \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 \mathbf{H}\|_{L^2}^2) \\
 &\leq C \|\nabla \mathbf{v}\|_{L^2}^2 \|\nabla \mathbf{H}\|_{L^2}^2 (1 + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2) + C e^{-2\beta_1 t} (1 + \|\nabla \mathbf{v}\|_{L^2}^8 + \|\nabla \mathbf{H}\|_{L^2}^8) + C \|\nabla \mathbf{H}\|_{L^2}^4 \\
 &+ \frac{2}{c^*} \frac{d}{dt} \int (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} dx + \frac{1}{2c^*} e^{-2\beta_1 t} (\|\sqrt{\rho_0} \mathbf{v}_0\|_{L^2}^2 + \|\mathbf{H}_0\|_{L^2}^2). \tag{22}
 \end{aligned}$$

Next, multiplying (22) by $e^{2\gamma t}$ with $\gamma \triangleq \min\{\frac{1}{4}, \frac{\beta_1}{4}, \frac{3\sigma}{8}\}$, and integrating the resulting inequality over $(0, t)$, we infer from (12) that

$$\begin{aligned}
 &e^{2\gamma t} \left(\|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \times \mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}\|_{L^2}^2 \right) \\
 &+ (1 - 2\gamma) \int_0^t e^{2\gamma s} \left(\|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \times \mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} \mathbf{v}_t\|_{L^2}^2 + \frac{1}{4} \|\mathbf{H}_t\|_{L^2}^2 \right) ds \\
 &+ (1 - 2\gamma) \int_0^t e^{2\gamma s} \frac{1}{4C_\Omega} (\|\nabla^2 \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 \mathbf{H}\|_{L^2}^2) ds \\
 &\leq C \int_0^t e^{2\gamma s} \|\nabla \mathbf{v}\|_{L^2}^2 \|\nabla \mathbf{H}\|_{L^2}^2 (1 + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2) ds \\
 &+ C \int_0^t e^{2(\gamma - \beta_1)s} (1 + \|\nabla \mathbf{v}\|_{L^2}^8 + \|\nabla \mathbf{H}\|_{L^2}^8) ds + C \int_0^t \frac{d}{ds} \left(e^{2\gamma s} \int (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} dx \right) ds \\
 &+ C \int_0^t \gamma e^{2\gamma s} \int (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} dx ds + C \int_0^t e^{2\gamma s} \|\nabla \mathbf{H}\|_{L^2}^4 ds \\
 &+ \|\nabla \mathbf{v}_0\|_{L^2}^2 + \|\nabla \times \mathbf{H}_0\|_{L^2}^2 + C (\|\sqrt{\rho_0} \mathbf{v}_0\|_{L^2}^2 + \|\mathbf{H}_0\|_{L^2}^2). \tag{23}
 \end{aligned}$$

It should be noted that the boundedness of $\|\nabla \mathbf{v}\|_{L^\infty(0,T;L^2)}$ and $\|\nabla \mathbf{H}\|_{L^\infty(0,T;L^2)}$, for any $T > 0$, was shown in [14, Lemma 3.2.], which combined with the decay estimate (6) for $t > 1$, we arrive at

$$\begin{aligned} & \int_0^t e^{2\gamma s} \|\nabla \mathbf{v}\|_{L^2}^2 \|\nabla \mathbf{H}\|_{L^2}^2 (1 + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2) ds \\ & \leq \int_0^\infty e^{2\gamma s} \|\nabla \mathbf{v}\|_{L^2}^2 \|\nabla \mathbf{H}\|_{L^2}^2 (1 + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2) ds \\ & \leq C \int_0^\infty e^{2\gamma s - 2\sigma s} + e^{2\gamma s - 3\sigma s} ds \leq C, \\ & \int_0^t e^{2\gamma s} \|\nabla \mathbf{H}\|_{L^2}^4 + e^{2(\gamma - \beta_1)s} \|\nabla(\mathbf{v}, \mathbf{H})\|_{L^2}^8 ds \\ & \leq C \int_0^\infty e^{2\gamma s - 2\sigma s} + e^{2(\gamma - \beta_1)s - 4\sigma s} ds \leq C, \end{aligned}$$

where C is a constant that may be enlarged if necessary. Consequently, by Hölder's inequality, we get

$$\begin{aligned} & \int_0^t \gamma e^{2\gamma s} \int (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} dx ds \leq C \int_0^\infty 2\gamma e^{2\gamma s} \|\mathbf{H}(s)\|_{H^1}^2 \|\nabla \mathbf{v}(s)\|_{L^2} ds \\ & \leq C \int_0^\infty 2\gamma e^{2\gamma s - \frac{3}{2}\sigma s} ds < C, \\ & \int_0^t \frac{d}{ds} \left(e^{2\gamma s} \int (\mathbf{H} \cdot \nabla) \mathbf{H} \cdot \mathbf{v} dx \right) ds \leq C e^{2\gamma t} \|\mathbf{H}(t)\|_{H^1}^2 \|\nabla \mathbf{v}(t)\|_{L^2} \leq C e^{2\gamma t} e^{-\sigma t} e^{-\frac{1}{2}\sigma t} < C. \end{aligned}$$

Combining the above estimates with (23), we derive (15). Finally, the inequality $\|\nabla \mathbf{H}\|_2 \leq C \|\nabla \times \mathbf{H}\|_2$ imply (16), while the Poincaré inequality and the Sobolev embedding theorem yield (17). $\square$

The following lemma is concerned with the time-weighted L^2 -estimate of $\sqrt{\rho} \mathbf{v}_t$ and $\mathbf{H}_t$.

Lemma 4. *There exists a positive constant $\beta_3 \leq \beta_2$ depending only on μ , ν , ρ^* , and Ω such that*

$$\sup_{t \in \mathbb{R}_+} (\|\sqrt{t} \rho e^{\beta_3 t} \mathbf{v}_t\|_{L^2}^2 + \|\sqrt{t} e^{\beta_3 t} \mathbf{H}_t\|_{L^2}^2) + \int_0^\infty t e^{2\beta_3 t} (\|\nabla \mathbf{v}_t\|_{L^2}^2 + \|\nabla \mathbf{H}_t\|_{L^2}^2 + \|\mathbf{v}_t\|_{L^2}^2 + \|\mathbf{H}_t\|_{L^2}^2) dt \leq C. \quad (24)$$

Proof. Differentiating (1)₂ and (1)₃ with respect to t , testing the resulting equations by $t e^{2\beta t} \mathbf{v}_t$

and $te^{2\beta t}\mathbf{H}_t$, respectively, and then integrating over Ω , we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int te^{2\beta t} \rho |\mathbf{v}_t|^2 + te^{2\beta t} |\mathbf{H}_t|^2 dx + \int te^{2\beta t} (\mu |\nabla \mathbf{v}_t|^2 + \nu |\nabla \mathbf{H}_t|^2) dx \\
&= \frac{1}{2} \int e^{2\beta t} \rho |\mathbf{v}_t|^2 dx + \int \beta te^{2\beta t} \rho |\mathbf{v}_t|^2 dx - \int te^{2\beta t} \rho_t \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{v}_t dx - \int te^{2\beta t} \rho \mathbf{v}_t \cdot \nabla \mathbf{v} \cdot \mathbf{v}_t dx \\
&\quad - \int \rho \mathbf{v} \cdot \nabla |\sqrt{te^{\beta t}} \mathbf{v}_t|^2 dx - \int te^{2\beta t} (\mathbf{H}_t \cdot \nabla) \mathbf{v}_t \cdot \mathbf{H} dx + \frac{1}{2} \int e^{2\beta t} |\mathbf{H}_t|^2 dx + \int \beta te^{2\beta t} |\mathbf{H}_t|^2 dx \\
&\quad - \int te^{2\beta t} (\mathbf{H}_t \cdot \nabla) \mathbf{H}_t \cdot \mathbf{v} dx - \int te^{2\beta t} (\mathbf{v}_t \cdot \nabla) \mathbf{H} \cdot \mathbf{H}_t dx \triangleq \sum_{i=1}^{10} I_i.
\end{aligned} \tag{25}$$

By virtue of Hölder's inequality and (11), the terms I_1 , I_2 , I_7 , and I_8 satisfy

$$|I_1| = \frac{1}{2} e^{2\beta t} \|\sqrt{\rho} \mathbf{v}_t\|_{L^2}^2, \tag{26}$$

$$|I_2| \leq \beta \|\sqrt{t\rho} e^{\beta t} \mathbf{v}_t\|_{L^2}^2 \leq \frac{1}{16} \|\sqrt{te^{\beta t}} \nabla \mathbf{v}_t\|_{L^2}^2, \tag{27}$$

$$|I_7| = \frac{1}{2} \|e^{\beta t} \mathbf{H}_t\|_{L^2}^2, \tag{28}$$

$$|I_8| \leq \frac{1}{8} \|\sqrt{te^{\beta t}} \nabla \mathbf{H}_t\|_{L^2}^2. \tag{29}$$

For I_3 , using the continuity equation $(1)_1$ and integrating by parts, we have

$$\begin{aligned}
|I_3| &= \left| - \int te^{2\beta t} \rho_t \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{v}_t dx \right| = \left| \int te^{2\beta t} \operatorname{div}(\rho \mathbf{v}) \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{v}_t dx \right| \\
&= \left| - \int te^{2\beta t} \rho \mathbf{v} \cdot \nabla ((\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \mathbf{v}_t) dx \right| \\
&\leq \int te^{2\beta t} \rho |\mathbf{v}| (|\nabla \mathbf{v}|^2 |\mathbf{v}_t| + |\mathbf{v}| |\nabla^2 \mathbf{v}| |\mathbf{v}_t| + |\mathbf{v}| |\nabla \mathbf{v}| |\nabla \mathbf{v}_t|) dx \\
&\triangleq R_1 + R_2 + R_3.
\end{aligned} \tag{30}$$

Applying Hölder's inequality and Gagliardo–Nirenberg inequality, we get

$$\begin{aligned}
R_1 &\leq C \sqrt{te^{2\beta t}} \|\sqrt{t\rho} \mathbf{v}_t\|_{L^4} \|\mathbf{v}\|_{L^6} \|\nabla \mathbf{v}\|_{L^{\frac{24}{7}}}^2 \\
&\leq C \sqrt{te^{2\beta t}} \|\sqrt{t\rho} \mathbf{v}_t\|_{L^2}^{\frac{1}{4}} \|\sqrt{t\rho} \mathbf{v}_t\|_{L^6}^{\frac{3}{4}} \|\mathbf{v}\|_{L^6} \|\nabla \mathbf{v}\|_{L^{\frac{24}{7}}}^2 \\
&\leq C \sqrt{te^{\frac{5}{4}\beta t}} \|\sqrt{t\rho} \mathbf{v}_t\|_{L^2}^{\frac{1}{4}} \|\sqrt{te^{\beta t}} \mathbf{v}_t\|_{L^6}^{\frac{3}{4}} \|\mathbf{v}\|_{L^6} \|\nabla \mathbf{v}\|_{L^{\frac{24}{7}}}^2 \\
&\leq \frac{1}{16} \|\sqrt{te^{\beta t}} \nabla \mathbf{v}_t\|_{L^2}^2 + Ct^{\frac{4}{5}} e^{2\beta t} \|\sqrt{t\rho} \mathbf{v}_t\|_{L^2}^{\frac{2}{5}} \|\nabla \mathbf{v}\|_{L^2}^{\frac{8}{5}} \|\nabla \mathbf{v}\|_{L^{\frac{24}{7}}}^{\frac{16}{5}} \\
&\leq \frac{1}{16} \|\sqrt{te^{\beta t}} \nabla \mathbf{v}_t\|_{L^2}^2 + Ct^{\frac{4}{5}} e^{-\frac{2}{5}\beta t} \|\sqrt{t\rho} e^{\beta t} \mathbf{v}_t\|_{L^2}^{\frac{2}{5}} \|\nabla \mathbf{v}\|_{L^2}^{\frac{14}{5}} \|e^{\beta t} \nabla^2 \mathbf{v}\|_{L^2}^2.
\end{aligned}$$

Hence, for some $c > 0$ independent of t ,

$$R_1 \leq \frac{1}{16} \|\sqrt{te^{\beta t}} \nabla \mathbf{v}_t\|_{L^2}^2 + Ce^{-ct} \|\sqrt{te^{\beta t}} \nabla^2 \mathbf{v}\|_{L^2}^2 (1 + \|\sqrt{t\rho} e^{\beta t} \mathbf{v}_t\|_{L^2}^2). \tag{31}$$

By virtue of Hölder's inequality, Young's inequality, and (16), we find that

$$\begin{aligned} R_2 &\leq C \|\sqrt{t}e^{\beta t} \nabla^2 \mathbf{v}\|_{L^2} \|\sqrt{t}\rho e^{\beta t} \mathbf{v}_t\|_{L^6} \|\mathbf{v}\|_{L^6}^2 \\ &\leq \frac{1}{16} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + C \|\nabla \mathbf{v}\|_{L^2}^4 \|\sqrt{t}e^{\beta t} \nabla^2 \mathbf{v}\|_{L^2}^2 \\ &\leq \frac{1}{16} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + C e^{-4\beta_2 t} \|\sqrt{t}e^{\beta t} \nabla^2 \mathbf{v}\|_{L^2}^2, \end{aligned} \quad (32)$$

$$\begin{aligned} R_3 &\leq C \|\sqrt{t}\rho e^{\beta t} \nabla \mathbf{v}_t\|_{L^2} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}\|_{L^6} \|\mathbf{v}\|_{L^6}^2 \\ &\leq \frac{1}{16} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + C \|\nabla \mathbf{v}\|_{L^2}^4 \|\sqrt{t}e^{\beta t} \nabla^2 \mathbf{v}\|_{L^2}^2 \\ &\leq \frac{1}{16} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + C e^{-4\beta_2 t} \|\sqrt{t}e^{\beta t} \nabla^2 \mathbf{v}\|_{L^2}^2. \end{aligned} \quad (33)$$

The remaining terms I_i ($i = 4, 5, 6, 9, 10$) are controlled by using (11), the Gagliardo–Nirenberg inequality, Young's inequality, and Hölder's inequality, which leads to

$$\begin{aligned} |I_4| &\leq C \|\sqrt{t}\rho e^{\beta t} \mathbf{v}_t\|_{L^4}^2 \|\nabla \mathbf{v}\|_{L^2} \\ &\leq C \|\sqrt{t}\rho e^{\beta t} \mathbf{v}_t\|_{L^2}^{\frac{1}{2}} \|\sqrt{t}\rho e^{\beta t} \mathbf{v}_t\|_{L^6}^{\frac{3}{2}} \|\nabla \mathbf{v}\|_{L^2} \\ &\leq \frac{1}{16} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + C \|\sqrt{t}\rho e^{\beta t} \mathbf{v}_t\|_{L^2}^2 \|\nabla \mathbf{v}\|_{L^2}^4, \end{aligned} \quad (34)$$

$$\begin{aligned} |I_5| &\leq C \|\mathbf{v}\|_{L^\infty} \|\sqrt{t}\rho e^{\beta t} \mathbf{v}_t\|_{L^2} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2} \\ &\leq \frac{1}{16} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + C \|\sqrt{t}\rho e^{\beta t} \mathbf{v}_t\|_{L^2}^2 \|\mathbf{v}\|_{L^\infty}^2, \end{aligned} \quad (35)$$

$$\begin{aligned} |I_6| &\leq \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2} \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^3} \|\mathbf{H}\|_{L^6} \\ &\leq C \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2} \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^2}^{\frac{1}{2}} \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{H}\|_{L^2} \\ &\leq \frac{1}{16} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + \frac{1}{8} \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2}^2 + C \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^2}^2 \|\nabla \mathbf{H}\|_{L^2}^4, \end{aligned} \quad (36)$$

$$\begin{aligned} |I_9| &\leq \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2} \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^3} \|\mathbf{v}\|_{L^6} \\ &\leq C \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2} \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^2}^{\frac{1}{2}} \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2} \\ &\leq \frac{1}{8} \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2}^2 + C \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^2}^2 \|\nabla \mathbf{v}\|_{L^2}^4, \end{aligned} \quad (37)$$

$$\begin{aligned} |I_{10}| &\leq \|\sqrt{t}e^{\beta t} \mathbf{v}_t\|_{L^6} \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^3} \|\nabla \mathbf{H}\|_{L^2} \\ &\leq C \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2} \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^2}^{\frac{1}{2}} \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{H}\|_{L^2} \\ &\leq \frac{1}{16} \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + \frac{1}{8} \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2}^2 + C \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^2}^2 \|\nabla \mathbf{H}\|_{L^2}^4. \end{aligned} \quad (38)$$

Substituting (26)–(38) into (25) gives

$$\frac{d}{dt} Y(t) + D(t) \leq CA(t)Y(t) + CB(t), \quad (39)$$

where

$$\begin{aligned} Y(t) &\triangleq \|\sqrt{t}\rho e^{\beta t} \mathbf{v}_t\|_{L^2}^2 + \|\sqrt{t}e^{\beta t} \mathbf{H}_t\|_{L^2}^2, \\ D(t) &\triangleq \|\sqrt{t}e^{\beta t} \nabla \mathbf{v}_t\|_{L^2}^2 + \|\sqrt{t}e^{\beta t} \nabla \mathbf{H}_t\|_{L^2}^2, \\ A(t) &\triangleq e^{-ct} \|\sqrt{t}e^{\beta t} \nabla^2 \mathbf{v}\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^4 + \|\mathbf{v}\|_{L^\infty}^2 + \|\nabla \mathbf{H}\|_{L^2}^4, \\ B(t) &\triangleq e^{-ct} \|\sqrt{t}e^{\beta t} \nabla^2 \mathbf{v}\|_{L^2}^2 + \frac{1}{2}e^{2\beta t} \|\sqrt{\rho} \mathbf{v}_t\|_{L^2}^2 + \frac{1}{2} \|e^{\beta t} \mathbf{H}_t\|_{L^2}^2. \end{aligned}$$

By virtue of Lemma 3, (8), and (16), provided that $0 < \beta \leq \beta_2$, we have

$$\int_0^\infty A(t) + B(t) dt \leq C.$$

Choosing $\beta_3 = \beta \leq \beta_2$, integrating (39) over $[0, \infty)$, and then applying Gronwall's inequality, we conclude that (24) holds. $\square$

Finally, we will use Lemmas 2–4 to establish higher-order norm estimates.

Lemma 5. *There exists a positive constant $\beta_4 < \beta_3$ depending only on μ, ν, ρ^* , and Ω such that*

$$\|\sqrt{t}e^{\beta_3 t} (\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P)\|_{L^p(\mathbb{R}_+; L^s)} \leq C \quad 2 \leq s \leq 6 \quad \text{and} \quad p = \frac{4s}{3s-6}, \quad (40)$$

$$\|e^{\beta_4 t} (\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H})\|_{L^1(\mathbb{R}_+; L^r)} \leq C \quad \text{for some } r \in (3, 6], \quad (41)$$

$$\|e^{\beta_4 t} (\nabla \mathbf{v}, \nabla \mathbf{H})\|_{L^1(\mathbb{R}_+; L^\infty)} \leq C. \quad (42)$$

Proof. Step 1: Proof of (40). Multiplying (1)₂ and (1)₃ by $\sqrt{t}e^{\beta t}$, we arrive at

$$-\mu \Delta \sqrt{t}e^{\beta t} \mathbf{v} + \sqrt{t}e^{\beta t} \nabla P = -\sqrt{t}e^{\beta t} \rho \mathbf{v}_t - \sqrt{t}e^{\beta t} \rho \mathbf{v} \cdot \nabla \mathbf{v} - \sqrt{t}e^{\beta t} \frac{1}{2} \nabla |\mathbf{H}|^2 + \sqrt{t}e^{\beta t} (\mathbf{H} \cdot \nabla) \mathbf{H}, \quad (43)$$

$$-\nu \Delta \sqrt{t}e^{\beta t} \mathbf{H} = -\sqrt{t}e^{\beta t} \mathbf{H}_t + \sqrt{t}e^{\beta t} (\mathbf{H} \cdot \nabla) \mathbf{v} - \sqrt{t}e^{\beta t} (\mathbf{v} \cdot \nabla) \mathbf{H}. \quad (44)$$

For any $2 \leq s \leq 6$, the interpolation inequality implies that

$$\|\sqrt{t}e^{\beta t} (\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P)\|_{L^s} \leq \|\sqrt{t}e^{\beta t} (\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P)\|_{L^2}^{\frac{6-s}{2s}} \|\sqrt{t}e^{\beta t} (\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P)\|_{L^6}^{\frac{3s-6}{2s}}.$$

Thus it suffices to show that

$$\sqrt{t}e^{\beta t} (\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P) \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \cap L^2(\mathbb{R}_+; L^6(\Omega)).$$

Furthermore, the estimate (24) together with the Sobolev embedding theorem ensures that

$$\sqrt{t}e^{\beta t} (\sqrt{\rho} \mathbf{v}_t, \mathbf{H}_t) \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \cap L^2(\mathbb{R}_+; L^6(\Omega)). \quad (45)$$

Applying the Stokes and elliptic regularity theories to (43) and (44), we deduce from (45) that

$$\begin{aligned} \|\sqrt{t}e^{\beta t}(\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P)\|_{L^\infty(\mathbb{R}_+; L^2)} &\leq C + C\|\sqrt{t}e^{\beta t}\rho \mathbf{v} \cdot \nabla \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)} \\ &\quad + C\|\sqrt{t}e^{\beta t}\mathbf{H} \cdot \nabla \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^2)} \\ &\quad + C\|\sqrt{t}e^{\beta t}(\mathbf{H} \cdot \nabla) \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)} \\ &\quad + C\|\sqrt{t}e^{\beta t}(\mathbf{v} \cdot \nabla) \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^2)} \triangleq C + C \sum_{i=1}^4 J_i. \end{aligned} \quad (46)$$

Combining Lemma 2 with Hölder's, Gagliardo–Nirenberg, and Young's inequalities, we derive

$$\begin{aligned} J_1 &\leq C\|e^{\beta t}\mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^6)}\|\sqrt{t}\nabla \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^3)} \\ &\leq C\|e^{\beta t}\nabla \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)}\|\sqrt{t}\nabla \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)}^{\frac{1}{2}}\|\sqrt{t}\nabla^2 \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)}^{\frac{1}{2}} \\ &\leq C\|\sqrt{t}e^{\beta t}\nabla^2 \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)}^{\frac{1}{2}} \\ &\leq \varepsilon\|\sqrt{t}e^{\beta t}\nabla^2 \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)} + C, \\ J_2 &\leq \|e^{\beta t}\mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^6)}\|\sqrt{t}\nabla \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^3)} \\ &\leq C\|e^{\beta t}\nabla \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^2)}\left(C_1\|\sqrt{t}\nabla \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^2)}^{\frac{1}{2}}\|\sqrt{t}\nabla^2 \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^2)}^{\frac{1}{2}} + C_2\|\sqrt{t}\nabla \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^2)}\right) \\ &\leq \varepsilon\|\sqrt{t}e^{\beta t}\nabla^2 \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^2)} + C. \end{aligned}$$

Similarly,

$$J_3 + J_4 \leq \varepsilon\left(\|\sqrt{t}e^{\beta t}\nabla^2 \mathbf{H}\|_{L^\infty(\mathbb{R}_+; L^2)} + \|\sqrt{t}e^{\beta t}\nabla^2 \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)}\right) + C.$$

Inserting these bounds into (46) leads to

$$\|\sqrt{t}e^{\beta t}(\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P)\|_{L^\infty(\mathbb{R}_+; L^2)} < \infty. \quad (47)$$

We next turn to the $L^2(\mathbb{R}_+; L^6)$ -bound. Applying the well-known regularity theory to (43) and (44), we obtain from (45) that

$$\begin{aligned} \|\sqrt{t}e^{\beta t}(\nabla^2 \mathbf{v}, \nabla^2 \mathbf{H}, \nabla P)\|_{L^2(\mathbb{R}_+; L^6)} &\leq C + C\|\sqrt{t}e^{\beta t}\rho \mathbf{v} \cdot \nabla \mathbf{v}\|_{L^2(\mathbb{R}_+; L^6)} \\ &\quad + C\|\sqrt{t}e^{\beta t}\mathbf{H} \cdot \nabla \mathbf{H}\|_{L^2(\mathbb{R}_+; L^6)} + C\|\sqrt{t}e^{\beta t}(\mathbf{H} \cdot \nabla) \mathbf{v}\|_{L^2(\mathbb{R}_+; L^6)} \\ &\quad + C\|\sqrt{t}e^{\beta t}(\mathbf{v} \cdot \nabla) \mathbf{H}\|_{L^2(\mathbb{R}_+; L^6)} \triangleq C + C \sum_{i=1}^4 K_i. \end{aligned} \quad (48)$$

Using (16), (47), Hölder's inequality, and Gagliardo–Nirenberg inequality, we have

$$\begin{aligned} K_1 &\leq C\|\sqrt{t}e^{\frac{\beta}{2}t}\mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^\infty)}\|e^{\frac{\beta}{2}t}\nabla \mathbf{v}\|_{L^2(\mathbb{R}_+; L^6)} \\ &\leq C\|\sqrt{t}e^{\frac{\beta}{2}t}\mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^6)}^{\frac{1}{2}}\|\sqrt{t}e^{\beta t}\nabla \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^6)}^{\frac{1}{2}}\|e^{\frac{\beta}{2}t}\nabla \mathbf{v}\|_{L^2(\mathbb{R}_+; L^6)} \\ &\leq C\|\sqrt{t}e^{\frac{\beta}{2}t}\nabla \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)}^{\frac{1}{2}}\|\sqrt{t}e^{\frac{\beta}{2}t}\nabla^2 \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)}^{\frac{1}{2}}\|e^{\frac{\beta}{2}t}\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}_+; L^2)} \\ &\leq C\left(\|\sqrt{t}e^{\frac{\beta}{2}t}\nabla \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)} + \|\sqrt{t}e^{\frac{\beta}{2}t}\nabla^2 \mathbf{v}\|_{L^\infty(\mathbb{R}_+; L^2)}\right) \leq C. \end{aligned}$$

Similarly,

$$\begin{aligned} K_2 + K_3 + K_4 &\leq C \left(\|\sqrt{t}e^{\frac{\beta}{2}t}\nabla \mathbf{v}\|_{L^\infty(\mathbb{R}_+;L^2)} + \|\sqrt{t}e^{\frac{\beta}{2}t}\nabla \mathbf{H}\|_{L^\infty(\mathbb{R}_+;L^2)} \right) \\ &\quad + C \left(\|\sqrt{t}e^{\beta t}\nabla^2 \mathbf{v}\|_{L^\infty(\mathbb{R}_+;L^2)} + \|\sqrt{t}e^{\beta t}\nabla^2 \mathbf{H}\|_{L^\infty(\mathbb{R}_+;L^2)} \right) \leq C. \end{aligned}$$

Substituting the above estimates into (48) and combining these results with (47), we obtain (40).

Step 2: Proof of (41). Choosing small positive parameters $\beta_4, \delta > 0$ such that $\beta_4 + \delta < \beta_3$, and splitting $[0, \infty)$ into $[0, 1]$ and $[1, \infty)$, it holds that

$$\begin{aligned} \int_0^\infty e^{\beta_4 t} (\|\nabla^2 \mathbf{v}\|_{L^r} + \|\nabla^2 \mathbf{H}\|_{L^r}) dt &= \int_0^1 e^{\beta_4 t} (\|\nabla^2 \mathbf{v}\|_{L^r} + \|\nabla^2 \mathbf{H}\|_{L^r}) dt \\ &\quad + \int_1^\infty e^{\beta_4 t} (\|\nabla^2 \mathbf{v}\|_{L^r} + \|\nabla^2 \mathbf{H}\|_{L^r}) dt. \end{aligned}$$

On the one hand, for $[0, 1]$, taking $p > 2$ in (40), so that $s = r \in (3, 6)$ and $\alpha = \frac{1}{2}$, we have $\alpha p > \frac{1}{2}$ and $\alpha p^* < 1$. Hence, by Hölder's inequality and $\beta_4 < \beta_2$, we have

$$\begin{aligned} \int_0^1 e^{\beta_4 t} (\|\nabla^2 \mathbf{v}\|_{L^r} + \|\nabla^2 \mathbf{H}\|_{L^r}) dt &= \int_0^1 e^{\beta_4 t} t^{-\alpha} t^\alpha \|\nabla^2 \mathbf{v}\|_{L^r} dt + \int_0^1 e^{\beta_4 t} t^{-\alpha} t^\alpha \|\nabla^2 \mathbf{H}\|_{L^r} dt \\ &\leq \left(\int_0^1 t^{-\alpha p^*} dt \right)^{1/p^*} \left(\int_0^1 t^{\alpha p} \|e^{\beta_4 t} \nabla^2 \mathbf{v}\|_{L^r}^p dt \right)^{1/p} \\ &\quad + \left(\int_0^1 t^{-\alpha p^*} dt \right)^{1/p^*} \left(\int_0^1 t^{\alpha p} \|e^{\beta_4 t} \nabla^2 \mathbf{H}\|_{L^r}^p dt \right)^{1/p} \leq C. \end{aligned}$$

It should be noted that for $p = 2$ and $s = 6$, one only needs to take $\alpha = \frac{1}{4}$, in which case the above still holds. On the other hand, combining $e^{\beta_4 t} = e^{(\beta_4 + \delta)t} e^{-\delta t}$ with (40), we arrive at

$$\begin{aligned} \int_1^\infty e^{(\beta_4 + \delta)t - \delta t} (\|\nabla^2 \mathbf{v}\|_{L^r} + \|\nabla^2 \mathbf{H}\|_{L^r}) dt &\leq \left(\int_1^\infty e^{-\delta p^* t} dt \right)^{1/p^*} \left(\int_1^\infty e^{(\beta_4 + \delta)pt} \|\nabla^2 \mathbf{v}\|_{L^r}^p dt \right)^{1/p} \\ &\quad + \left(\int_1^\infty e^{-\delta p^* t} dt \right)^{1/p^*} \left(\int_1^\infty e^{(\beta_4 + \delta)pt} \|\nabla^2 \mathbf{H}\|_{L^r}^p dt \right)^{1/p} \\ &\leq C \left(\|e^{(\beta_4 + \delta)t} \nabla^2 \mathbf{v}\|_{L^p(\mathbb{R}_+;L^r)} + \|e^{(\beta_4 + \delta)t} \nabla^2 \mathbf{H}\|_{L^p(\mathbb{R}_+;L^r)} \right) \\ &\leq C \left(\|e^{\beta_3 t} \nabla^2 \mathbf{v}\|_{L^p(\mathbb{R}_+;L^r)} + \|e^{\beta_3 t} \nabla^2 \mathbf{H}\|_{L^p(\mathbb{R}_+;L^r)} \right) \leq C. \end{aligned}$$

Combining the estimates on the two time intervals establishes (41).

Step 3: Proof of (42). It follows from the Gagliardo–Nirenberg inequality that

$$\begin{aligned} \|e^{\beta_4 t} \nabla \mathbf{v}\|_{L^1(\mathbb{R}_+;L^\infty)} + \|e^{\beta_4 t} \nabla \mathbf{H}\|_{L^1(\mathbb{R}_+;L^\infty)} &\leq C \left(\|e^{\beta_4 t} \nabla \mathbf{v}\|_{L^1(\mathbb{R}_+;L^2)} + \|e^{\beta_4 t} \nabla^2 \mathbf{v}\|_{L^1(\mathbb{R}_+;L^r)} \right. \\ &\quad \left. + \|e^{\beta_4 t} \nabla \mathbf{H}\|_{L^1(\mathbb{R}_+;L^2)} + \|e^{\beta_4 t} \nabla^2 \mathbf{H}\|_{L^1(\mathbb{R}_+;L^r)} \right), \end{aligned}$$

which together with (16) and (41) leads to (42). $\square$

2 Proof of Theorem 1

In this section we work in Lagrangian coordinates to prove the stability result stated in Theorem 1. Let $X(t, y) \mapsto x$ denote the flow map associated with $\mathbf{v}$, with

$$\frac{dX}{dt} = \mathbf{v}(t, X(t, y)) \quad \text{in } \mathbb{R}_+ \times \Omega, \quad X(0, y) = y \quad \text{in } \Omega. \quad (49)$$

Integrating (49) over $(0, t)$ gives

$$x = X(t, y) = y + \int_0^t \mathbf{v}(t', X(t', y)) dt'.$$

The corresponding Lagrangian unknowns are

$$\begin{cases} \eta \triangleq \rho(t, X(t, y)), & \text{and} & \mathbf{u} \triangleq \mathbf{v}(t, X(t, y)), \\ \mathbf{d} \triangleq \mathbf{H}(t, X(t, y)), & \text{and} & Q \triangleq (P + \frac{1}{2}|\mathbf{H}|^2)(t, X(t, y)). \end{cases} \quad (50)$$

Next, we define the inverse deformation gradient matrix $A(t)$ as

$$A(t, y) \triangleq \left(\frac{dX}{dy} \right)^{-1} = \left(Id + \int_0^t \nabla_y \mathbf{u}(t', y) dt' \right)^{-1} = \left[\text{cof} \left(Id + \int_0^t \nabla_y \mathbf{u}(t', y) dt' \right) \right]^T, \quad (51)$$

where cof denotes the cofactor matrix. Although the matrix A need not be close to Id , it is invertible and uniformly bounded in time. Thus, in the (t, y) -coordinates, operators ∇ , div , and Δ translate into

$$\begin{aligned} \nabla_{\mathbf{u}} &\triangleq \nabla_x = A^T \nabla_y, & \text{curl}_{\mathbf{u}} &\triangleq A^T \text{curl}_y, \\ \text{div}_{\mathbf{u}} &\triangleq A^T : \nabla_y = \text{div}_y(A \cdot), & \Delta_{\mathbf{u}} &\triangleq \text{div}_y(A A^T \nabla_y). \end{aligned} \quad (52)$$

For simplicity, we present the system in the normalized case $\mu = \nu = 1$. Consequently, in terms of the variables $(\eta, \mathbf{u}, \mathbf{d}, Q)$, system (1) reduces to

$$\begin{cases} \eta_t = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \eta \mathbf{u}_t - \Delta_{\mathbf{u}} \mathbf{u} + \nabla_{\mathbf{u}} Q = \mathbf{d} \cdot \nabla_{\mathbf{u}} \mathbf{d} & \text{in } \mathbb{R}_+ \times \Omega, \\ \mathbf{d}_t - \Delta_{\mathbf{u}} \mathbf{d} - \mathbf{d} \cdot \nabla_{\mathbf{u}} \mathbf{u} = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \text{div}_{\mathbf{u}} \mathbf{u} = 0, \quad \text{div}_{\mathbf{u}} \mathbf{d} = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases} \quad (53)$$

It follows from the first equation in (53) that the density is time-independent in Lagrangian coordinates, i.e., $\eta(t, y) = \eta(0, y) = \rho_0(y)$.

By the decay estimates established in Section 1, the following time-weighted bounds hold

for both solutions ($i = 1, 2$):

$$\left\{ \begin{array}{l} e^{\beta_0 t}(\sqrt{\rho^i} \mathbf{v}^i, \mathbf{H}^i) \in L^2(\mathbb{R}_+; L^2), \\ e^{\beta_0 t}(\nabla \mathbf{v}^i, \nabla \mathbf{H}^i) \in L^1(\mathbb{R}_+; L^\infty) \cap L^4(\mathbb{R}_+; L^3) \cap L^2(\mathbb{R}_+; L^6), \\ \sqrt{t} e^{\beta_0 t}(\mathbf{v}^i, \mathbf{H}^i) \in L^1(\mathbb{R}_+; L^\infty) \cap L^\infty(\mathbb{R}_+; L^\infty), \\ \sqrt{t} e^{\beta_0 t}(\nabla^2 \mathbf{v}^i, \nabla^2 \mathbf{H}^i, \nabla P^i) \in L^2(\mathbb{R}_+; L^6), \\ \sqrt{t} e^{\beta_0 t}(\mathbf{v}_t^i, \mathbf{H}_t^i) \in L^{\frac{4}{3}}(\mathbb{R}_+; L^6), \\ e^{\beta_0 t}(\mathbf{v}^i, \mathbf{H}^i) \in L^1(\mathbb{R}_+; L^6) \cap L^\infty(\mathbb{R}_+; L^6), \\ \sqrt{t} e^{\beta_0 t}(\nabla \mathbf{v}^i, \nabla \mathbf{H}^i) \in L^2(\mathbb{R}_+; L^\infty), \\ e^{\beta_0 t}(\nabla^2 \mathbf{v}^i, \nabla^2 \mathbf{H}^i) \in L^1(\mathbb{R}_+; L^r) \quad \text{for some } r \in (3, 6], \end{array} \right. \quad (54)$$

where $0 < \beta_0 \leq \beta_4$ depends only on μ, ν, ρ^* , and Ω .

Denoting by $\delta \mathbf{u} \triangleq \mathbf{u}^2 - \mathbf{u}^1$, $\delta \mathbf{d} \triangleq \mathbf{d}^2 - \mathbf{d}^1$, and $\delta \rho \triangleq \rho^2 - \rho^1$, we obtain from (52) and (53) that

$$\left\{ \begin{array}{l} \rho_0^1 \delta \mathbf{u}_t - \Delta_{\mathbf{u}^1} \delta \mathbf{u} + \nabla_{\mathbf{u}^1} \delta Q = (\Delta_{\mathbf{u}^2} - \Delta_{\mathbf{u}^1}) \mathbf{u}^2 - (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) Q^2 + \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{d} \\ \quad + \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{d}^1 + \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{d}^1 - \delta \rho_0 \mathbf{u}_t^2, \\ \delta \mathbf{d}_t - \Delta_{\mathbf{u}^1} \delta \mathbf{d} = (\Delta_{\mathbf{u}^2} - \Delta_{\mathbf{u}^1}) \mathbf{d}^2 + \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{u}^1 + \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{u} + \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{u}^1, \\ \operatorname{div}_{\mathbf{u}^1} \delta \mathbf{u} = (\operatorname{div}_{\mathbf{u}^1} - \operatorname{div}_{\mathbf{u}^2}) \mathbf{u}^2, \quad \operatorname{div}_{\mathbf{u}^1} \delta \mathbf{d} = (\operatorname{div}_{\mathbf{u}^1} - \operatorname{div}_{\mathbf{u}^2}) \mathbf{d}^2, \\ \delta \mathbf{u}|_{t=0} = \delta \mathbf{v}_0, \quad \delta \mathbf{d}|_{t=0} = \delta \mathbf{H}_0, \end{array} \right. \quad (55)$$

with $\delta \rho|_{t=0} = \rho_0^2 - \rho_0^1$.

Next, we decompose $\delta \mathbf{u}$ as

$$\delta \mathbf{u} = \widehat{\mathbf{z}} + \widetilde{\mathbf{z}},$$

where the auxiliary field $\widehat{\mathbf{z}}$ is chosen to account for the divergence constraints:

$$\operatorname{div}_{\mathbf{u}^1} \widehat{\mathbf{z}} = (\operatorname{div}_{\mathbf{u}^1} - \operatorname{div}_{\mathbf{u}^2}) \mathbf{u}^2 = -\delta A^T : \nabla \mathbf{u}^2 = -\operatorname{div}(\delta A \mathbf{u}^2), \quad \widehat{\mathbf{z}}|_{\partial \Omega} = 0. \quad (56)$$

Here $\delta A = A^2 - A^1$, where A^i denotes the inverse deformation gradient matrix associated with the i -th solution. Meanwhile, the estimates for A have already been established in [17, pp. 908], namely,

$$\|t^{-\frac{1}{2}} \delta A\|_2 \leq C \left(\int_0^t \|\nabla \delta \mathbf{u}\|_2^2 d\tau \right)^{\frac{1}{2}}, \quad \text{for } t > 0. \quad (57)$$

Since $\delta A|_{t=0} = 0$ by (51), we have $\widehat{\mathbf{z}}|_{t=0} = 0$. Moreover, the results of [12, Lemma A.3]

and [17, Lemma 2] imply

$$e^{\beta_0 t} \|\widehat{\mathbf{z}}(t)\|_{L^2} \leq f_{1,\infty}(t) \left(\int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}}, \quad (58)$$

$$e^{\beta_0 t} \|\nabla \widehat{\mathbf{z}}(t)\|_{L^2} \leq f_2(t) \left(\int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}}, \quad (59)$$

$$e^{\beta_0 t} \|\widehat{\mathbf{z}}_t(t)\|_{L^2} \leq f_{\frac{4}{3}}(t) \left(\int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} + f_4(t) \|\nabla \delta \mathbf{u}\|_{L^2}. \quad (60)$$

Using the decomposition, for $i = 1, 2$, system (55) takes the following form:

$$\begin{cases} \rho_0^1 \widetilde{\mathbf{z}}_t - \Delta_{\mathbf{u}^1} \widetilde{\mathbf{z}} + \nabla_{\mathbf{u}^1} \delta Q = (\Delta_{\mathbf{u}^2} - \Delta_{\mathbf{u}^1}) \mathbf{u}^2 - (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) Q^2 + \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{d} \\ + \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{d}^1 + \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{d}^1 - \delta \rho_0 \mathbf{u}_t^2 - \rho_0^1 \widehat{\mathbf{z}}_t + \Delta_{\mathbf{u}^1} \widehat{\mathbf{z}}, \\ \delta \mathbf{d}_t - \Delta_{\mathbf{u}^1} \delta \mathbf{d} = (\Delta_{\mathbf{u}^2} - \Delta_{\mathbf{u}^1}) \mathbf{d}^2 + \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{u}^1 + \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{u} + \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{u}^1, \\ \operatorname{div}_{\mathbf{u}^1} \widetilde{\mathbf{z}} = 0, \quad \operatorname{div}_{\mathbf{u}^1} \delta \mathbf{d} = (\operatorname{div}_{\mathbf{u}^1} - \operatorname{div}_{\mathbf{u}^2}) \mathbf{d}^2, \\ \widetilde{\mathbf{z}}|_{\partial\Omega} = 0, \quad (\delta \mathbf{d} \cdot \mathbf{n})|_{\partial\Omega} = (\nabla_{\mathbf{u}^i} \times \mathbf{d}^i) \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}, \\ \widetilde{\mathbf{z}}|_{t=0} = \delta \mathbf{v}_0, \quad \delta \mathbf{d}|_{t=0} = \delta \mathbf{H}_0. \end{cases} \quad (61)$$

We are now in a position to prove Theorem 1 by considering separately the non-vacuum and vacuum cases. We first consider the non-vacuum cases, in which case the inclusion $\operatorname{supp} \delta \rho_0 \subset \operatorname{supp} \rho_0^2$ holds true. Testing (61)₁ against $\widetilde{\mathbf{z}}$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \rho_0^1 |\widetilde{\mathbf{z}}|^2 dy + \int |\nabla_{\mathbf{u}^1} \widetilde{\mathbf{z}}|^2 dy &= \int (\Delta_{\mathbf{u}^2} - \Delta_{\mathbf{u}^1}) \mathbf{u}^2 \cdot \widetilde{\mathbf{z}} dy - \int (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) Q^2 \cdot \widetilde{\mathbf{z}} dy \\ &+ \int \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{d} \cdot \widetilde{\mathbf{z}} dy + \int \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{d}^1 \cdot \widetilde{\mathbf{z}} dy \\ &+ \int \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{d}^1 \cdot \widetilde{\mathbf{z}} dy - \int \rho_0^1 \widehat{\mathbf{z}}_t \cdot \widetilde{\mathbf{z}} dy \\ &+ \int \Delta_{\mathbf{u}^1} \widehat{\mathbf{z}} \cdot \widetilde{\mathbf{z}} dy - \int \delta \rho_0 \mathbf{u}_t^2 \cdot \widetilde{\mathbf{z}} dy, \end{aligned} \quad (62)$$

where the pressure term vanishes since

$$\int (\nabla_{\mathbf{u}^1} \delta Q) \cdot \widetilde{\mathbf{z}} dy = - \int \operatorname{div}_{\mathbf{u}^1} \widetilde{\mathbf{z}} \delta Q dy = 0.$$

Likewise, pairing (61)₂ with $\delta \mathbf{d}$ yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\delta \mathbf{d}|^2 dy + \int (-\Delta_{\mathbf{u}^2} \mathbf{d}^2 + \Delta_{\mathbf{u}^1} \mathbf{d}^1) \cdot \delta \mathbf{d} dy &= \int \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{u}^1 \cdot \delta \mathbf{d} dy + \int \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{u} \cdot \delta \mathbf{d} dy \\ &+ \int \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{u}^1 \cdot \delta \mathbf{d} dy. \end{aligned} \quad (63)$$

By virtue of $\operatorname{div}_{\mathbf{u}^i} \mathbf{d}^i = 0$ and $\operatorname{curl}_{\mathbf{u}^i} \mathbf{d}^i \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}$, $i = 1, 2$, we infer that

$$\begin{aligned} \int (-\Delta_{\mathbf{u}^2} \mathbf{d}^2 + \Delta_{\mathbf{u}^1} \mathbf{d}^1) \cdot \delta \mathbf{d} dy &= \int |\operatorname{curl}_{\mathbf{u}^1} \delta \mathbf{d}|^2 dy + \int \operatorname{curl}_{\mathbf{u}^2} \mathbf{d}^2 \cdot \operatorname{curl}_{\mathbf{u}^2} \delta \mathbf{d} dy \\ &\quad - \int \operatorname{curl}_{\mathbf{u}^1} \mathbf{d}^2 \cdot \operatorname{curl}_{\mathbf{u}^1} \delta \mathbf{d} dy \\ &= \int |\operatorname{curl}_{\mathbf{u}^1} \delta \mathbf{d}|^2 dy + \int (\operatorname{curl}_{\mathbf{u}^2} - \operatorname{curl}_{\mathbf{u}^1}) \mathbf{d}^2 \cdot \operatorname{curl}_{\mathbf{u}^2} \delta \mathbf{d} dy \\ &\quad + \int \operatorname{curl}_{\mathbf{u}^1} \mathbf{d}^2 \cdot (\operatorname{curl}_{\mathbf{u}^2} - \operatorname{curl}_{\mathbf{u}^1}) \delta \mathbf{d} dy. \end{aligned}$$

Note that

$$\begin{aligned} \int \operatorname{curl}_{\mathbf{u}^i} \operatorname{curl}_{\mathbf{u}^i} \mathbf{d}^i \cdot \delta \mathbf{d} dy &= \int \operatorname{curl}_{\mathbf{u}^i} \mathbf{d}^i \cdot \operatorname{curl}_{\mathbf{u}^i} \delta \mathbf{d} dy + \int_{\partial\Omega} \mathbf{n} \times \operatorname{curl}_{\mathbf{u}^i} \mathbf{d}^i \cdot \delta \mathbf{d} dy \\ &= \int \operatorname{curl}_{\mathbf{u}^i} \mathbf{d}^i \cdot \operatorname{curl}_{\mathbf{u}^i} \delta \mathbf{d} dy. \end{aligned}$$

Consequently, (63) can be rewritten as

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\delta \mathbf{d}|^2 dy + \int |\operatorname{curl}_{\mathbf{u}^1} \delta \mathbf{d}|^2 dy \\ &= \int \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{u}^1 \cdot \delta \mathbf{d} dy + \int \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{u} \cdot \delta \mathbf{d} dy + \int \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{u}^1 \cdot \delta \mathbf{d} dy \\ &\quad - \int (\operatorname{curl}_{\mathbf{u}^2} - \operatorname{curl}_{\mathbf{u}^1}) \mathbf{d}^2 \cdot \operatorname{curl}_{\mathbf{u}^2} \delta \mathbf{d} dy - \int \operatorname{curl}_{\mathbf{u}^1} \mathbf{d}^2 \cdot (\operatorname{curl}_{\mathbf{u}^2} - \operatorname{curl}_{\mathbf{u}^1}) \delta \mathbf{d} dy. \end{aligned} \quad (64)$$

Summing (62) and (64), we arrive at

$$\frac{1}{2} \frac{d}{dt} \int (\rho_0^1 |\tilde{\mathbf{z}}|^2 + |\delta \mathbf{d}|^2) dy + \int |\nabla_{\mathbf{u}^1} \tilde{\mathbf{z}}|^2 dy + \int |\operatorname{curl}_{\mathbf{u}^1} \delta \mathbf{d}|^2 dy \triangleq \sum_{k=1}^{13} I_k. \quad (65)$$

For any $\varepsilon > 0$, by (54), (57), and Young's inequality, I_1 satisfy

$$\begin{aligned} |I_1| &= \left| \int (\Delta_{\mathbf{u}^2} - \Delta_{\mathbf{u}^1}) \mathbf{u}^2 \cdot \tilde{\mathbf{z}} dy \right| = \left| \int \operatorname{div} \left((\delta A (A^2)^T + A^1 (\delta A)^T) \nabla \mathbf{u}^2 \right) \cdot \tilde{\mathbf{z}} dy \right| \\ &= \left| - \int (\delta A (A^2)^T + A^1 (\delta A)^T) \nabla \mathbf{u}^2 \cdot \nabla \tilde{\mathbf{z}} dy \right| \\ &\leq C \|t^{-\frac{1}{2}} \delta A\|_{L^2} \|\sqrt{t} \nabla \mathbf{u}^2\|_{L^\infty} \|\nabla \tilde{\mathbf{z}}\|_{L^2} \\ &\leq \varepsilon \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + C(\varepsilon) \|\sqrt{t} \nabla \mathbf{u}^2\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau. \end{aligned} \quad (66)$$

The pressure term is handled as follows:

$$\begin{aligned} |I_2| &= \left| - \int (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) Q^2 \cdot \tilde{\mathbf{z}} dy \right| \leq \|\delta A\|_{L^2} \|\nabla Q^2\|_{L^4} \|\tilde{\mathbf{z}}\|_{L^4} \leq \|t^{-\frac{1}{2}} \delta A\|_{L^2} \|\sqrt{t} \nabla Q^2\|_{L^4} \|\tilde{\mathbf{z}}\|_{L^4} \\ &\leq \varepsilon \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + C(\varepsilon) \|\sqrt{t} \nabla Q^2\|_{L^4}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau. \end{aligned} \quad (67)$$

The terms I_3 and I_{10} are estimated together. In view of $\operatorname{div}_{\mathbf{u}^2} \mathbf{d}^2 = 0$, (16), and (59), we obtain from Hölder's inequality and Young's inequality that

$$\begin{aligned}
 |I_3 + I_{10}| &= \left| \int \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{d} \cdot \tilde{\mathbf{z}} dy + \int \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{u} \cdot \delta \mathbf{d} dy \right| \\
 &= \left| - \int \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \tilde{\mathbf{z}} \cdot \delta \mathbf{d} dy + \int \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \delta \mathbf{u} \cdot \delta \mathbf{d} dy \right| \\
 &= \left| \int \mathbf{d}^2 \cdot \nabla_{\mathbf{u}^2} \hat{\mathbf{z}} \cdot \delta \mathbf{d} dy \right| \leq \|\nabla_{\mathbf{u}^2} \hat{\mathbf{z}}\|_{L^2} \|\delta \mathbf{d}\|_{L^4} \|\mathbf{d}^2\|_{L^4} \\
 &\leq \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) \|\nabla \mathbf{d}^2\|_{L^2}^2 \|\nabla_{\mathbf{u}^2} \hat{\mathbf{z}}\|_{L^2}^2 \\
 &\leq \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) e^{-2\beta_2 t} \|\nabla_{\mathbf{u}^2} \hat{\mathbf{z}}\|_{L^2}^2 \leq \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) f_1(t) \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau. \quad (68)
 \end{aligned}$$

Using (16), (52), (54), (57), and Hölder's inequality, we infer that

$$\begin{aligned}
 |I_4| &= \left| \int \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{d}^1 \cdot \tilde{\mathbf{z}} dy \right| \leq \int |\mathbf{d}^2| |\delta A| |\nabla \mathbf{d}^1| |\tilde{\mathbf{z}}| dy \\
 &\leq \|t^{-\frac{1}{2}} \delta A\|_{L^2} \|\sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty} \|\mathbf{d}^2\|_{L^4} \|\tilde{\mathbf{z}}\|_{L^4} \\
 &\leq \varepsilon \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + C(\varepsilon) \|\sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty}^2 \|\nabla \mathbf{d}^2\|_{L^2}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \\
 &\leq \varepsilon \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + C(\varepsilon) e^{-2\beta_2 t} \|\sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \\
 &\leq \varepsilon \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + C(\varepsilon) \|\sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau, \quad (69)
 \end{aligned}$$

$$\begin{aligned}
 |I_{11}| &= \left| \int \mathbf{d}^2 \cdot (\nabla_{\mathbf{u}^2} - \nabla_{\mathbf{u}^1}) \mathbf{u}^1 \cdot \delta \mathbf{d} dy \right| \leq \int |\mathbf{d}^2| |\delta A| |\nabla \mathbf{u}^1| |\delta \mathbf{d}| dy \\
 &\leq \|t^{-\frac{1}{2}} \delta A\|_{L^2} \|\sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty} \|\mathbf{d}^2\|_{L^4} \|\delta \mathbf{d}\|_{L^4} \\
 &\leq \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) \|\sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty}^2 \|\nabla \mathbf{d}^2\|_{L^2}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \\
 &\leq \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) e^{-2\beta_2 t} \|\sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \\
 &\leq \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) \|\sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau, \quad (70)
 \end{aligned}$$

$$\begin{aligned}
 |I_{12}| &= \left| - \int (\operatorname{curl}_{\mathbf{u}^2} - \operatorname{curl}_{\mathbf{u}^1}) \mathbf{d}^2 \cdot \operatorname{curl}_{\mathbf{u}^2} \delta \mathbf{d} dy \right| \\
 &\leq \int |\delta A| |\operatorname{curl}_y \mathbf{d}^2| |A^2| |\operatorname{curl}_y \delta \mathbf{d}| dy \\
 &\leq C \|t^{-\frac{1}{2}} \delta A\|_{L^2} \|\operatorname{curl}_y \delta \mathbf{d}\|_{L^2} \|\sqrt{t} \operatorname{curl}_y \mathbf{d}^2\|_{L^\infty} \\
 &\leq \varepsilon \|\operatorname{curl}_y \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) \|\sqrt{t} \operatorname{curl}_y \mathbf{d}^2\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau, \quad (71)
 \end{aligned}$$

$$\begin{aligned}
|I_{13}| &= \left| - \int \operatorname{curl}_{\mathbf{u}^1} \mathbf{d}^2 \cdot (\operatorname{curl}_{\mathbf{u}^2} - \operatorname{curl}_{\mathbf{u}^1}) \delta \mathbf{d} \, dy \right| \leq \int |A^1| |\operatorname{curl}_y \mathbf{d}^2| |\delta A| |\operatorname{curl}_y \delta \mathbf{d}| \, dy \\
&\leq C \|t^{-\frac{1}{2}} \delta A\|_{L^2} \|\operatorname{curl}_y \delta \mathbf{d}\|_{L^2} \|\sqrt{t} \operatorname{curl}_y \mathbf{d}^2\|_{L^\infty} \\
&\leq \varepsilon \|\operatorname{curl}_y \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) \|\sqrt{t} \operatorname{curl}_y \mathbf{d}^2\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 \, d\tau.
\end{aligned} \tag{72}$$

Then, combining (16) with Hölder's, Gagliardo–Nirenberg, and Young's inequalities, we obtain

$$\begin{aligned}
|I_5| &= \left| \int \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{d}^1 \cdot \tilde{\mathbf{z}} \, dy \right| \leq \|\nabla_{\mathbf{u}^1} \mathbf{d}^1\|_{L^2} \|\delta \mathbf{d}\|_{L^4} \|\tilde{\mathbf{z}}\|_{L^4} \leq \|\nabla_{\mathbf{u}^1} \mathbf{d}^1\|_{L^2} \|\delta \mathbf{d}\|_{L^2}^{\frac{1}{4}} \|\nabla \delta \mathbf{d}\|_{L^2}^{\frac{3}{4}} \|\tilde{\mathbf{z}}\|_{L^4} \\
&\leq \varepsilon \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) e^{-2\beta_4 t} \|\delta \mathbf{d}\|_{L^2}^2,
\end{aligned} \tag{73}$$

$$\begin{aligned}
|I_9| &= \left| \int \delta \mathbf{d} \cdot \nabla_{\mathbf{u}^1} \mathbf{u}^1 \cdot \delta \mathbf{d} \, dy \right| \leq \|\nabla_{\mathbf{u}^1} \mathbf{u}^1\|_{L^2} \|\delta \mathbf{d}\|_{L^4}^2 \leq \|\nabla_{\mathbf{u}^1} \mathbf{u}^1\|_{L^2} \|\delta \mathbf{d}\|_{L^2}^{\frac{1}{2}} \|\nabla \delta \mathbf{d}\|_{L^2}^{\frac{3}{2}} \\
&\leq \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) \|\delta \mathbf{d}\|_{L^2}^2 \|\nabla_{\mathbf{u}^1} \mathbf{u}^1\|_{L^2}^4 \leq \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) e^{-2\beta_2 t} \|\delta \mathbf{d}\|_{L^2}^2.
\end{aligned} \tag{74}$$

Owing to $0 \leq \rho_0^1 \leq \rho^*$, Hölder's inequality and the Gagliardo–Nirenberg inequality imply

$$\begin{aligned}
|I_6| &= \left| - \int \rho_0^1 \tilde{\mathbf{z}}_t \cdot \tilde{\mathbf{z}} \, dy \right| \leq C \|\tilde{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} \left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^3} \leq C \|\tilde{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} (\rho_0^1)^{1/4} \|\tilde{\mathbf{z}}\|_{L^3} \\
&\leq C \|\tilde{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} \|\tilde{\mathbf{z}}\|_{L^6}^{\frac{1}{2}} \left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^{\frac{1}{2}} \leq C \|\tilde{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} \|\nabla \tilde{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^{\frac{1}{2}},
\end{aligned} \tag{75}$$

while integration by parts gives

$$|I_7| = \left| \int \Delta_{\mathbf{u}^1} \tilde{\mathbf{z}} \cdot \tilde{\mathbf{z}} \, dy \right| = \left| - \int \nabla_{\mathbf{u}^1} \tilde{\mathbf{z}} \cdot \nabla_{\mathbf{u}^1} \tilde{\mathbf{z}} \, dy \right| \leq \varepsilon \|\nabla_{\mathbf{u}^1} \tilde{\mathbf{z}}\|_{L^2}^2 + C(\varepsilon) \|\nabla_{\mathbf{u}^1} \tilde{\mathbf{z}}\|_{L^2}^2. \tag{76}$$

Finally, applying Hölder's inequality, the Sobolev embedding, and Young's inequality, we get

$$\begin{aligned}
|I_8| &= \left| \int \delta \rho_0 \mathbf{u}_t^2 \cdot \tilde{\mathbf{z}} \, dy \right| \leq \int \left| \frac{\delta \rho_0}{\sqrt{\rho_0^2}} \right| \left| \sqrt{\rho_0^2} \mathbf{u}_t^2 \right| |\tilde{\mathbf{z}}| \, dy \leq \left\| \sqrt{\rho_0^2} \mathbf{u}_t^2 \right\|_{L^2} \|\tilde{\mathbf{z}}\|_{L^4} \left\| \frac{\delta \rho_0}{\sqrt{\rho_0^2}} \right\|_{L^4} \\
&\leq \varepsilon \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + C(\varepsilon) \|\delta \rho_0\|_X^2 \left\| \sqrt{\rho_0^2} \mathbf{u}_t^2 \right\|_{L^2}^2.
\end{aligned} \tag{77}$$

Substituting (66)–(77) into (65), we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^2 + \|\delta \mathbf{d}\|_{L^2}^2 \right) + \|\nabla_{\mathbf{u}^1} \tilde{\mathbf{z}}\|_{L^2}^2 + \|\operatorname{curl}_{\mathbf{u}^1} \delta \mathbf{d}\|_{L^2}^2 \\
&\leq \varepsilon \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + \varepsilon \|\operatorname{curl}_y \delta \mathbf{d}\|_{L^2}^2 + \varepsilon \|\nabla \delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) \left(\|\sqrt{t} \nabla \mathbf{u}^2\|_{L^\infty}^2 + \|\sqrt{t} \operatorname{curl}_y \mathbf{d}^2\|_{L^\infty}^2 + \|\sqrt{t} \nabla Q^2\|_{L^4}^2 \right. \\
&\quad \left. + f_1(t) + \|\sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty}^2 + \|\sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty}^2 \right) \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 \, d\tau + C \|\tilde{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} \|\nabla \tilde{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^{\frac{1}{2}} \\
&\quad + C(\varepsilon) e^{-2\beta_2 t} \|\delta \mathbf{d}\|_{L^2}^2 + C(\varepsilon) \|\nabla_{\mathbf{u}^1} \tilde{\mathbf{z}}\|_{L^2}^2 + C(\varepsilon) \|\delta \rho_0\|_X^2 \left\| \sqrt{\rho_0^2} \mathbf{u}_t^2 \right\|_{L^2}^2.
\end{aligned} \tag{78}$$

Given some vector field b satisfying $\nabla_{\mathbf{u}^i} b = (A^i)^T \nabla b$ and $\operatorname{curl}_{\mathbf{u}^1} b = (A^1)^T \operatorname{curl}_y b$, we have

$$\begin{aligned}
\|(A^i)^{-1}\|_{L^\infty}^{-1} \|\nabla b\|_{L^2} &\leq \|\nabla_{\mathbf{u}^i} b\|_{L^2} \leq \|(A^i)^T\|_{L^\infty} \|\nabla b\|_{L^2}, \\
\|(A^1)^{-1}\|_{L^\infty}^{-1} \|\operatorname{curl}_y b\|_{L^2} &\leq \|\operatorname{curl}_{\mathbf{u}^1} b\|_{L^2} \leq \|(A^1)^T\|_{L^\infty} \|\operatorname{curl}_y b\|_{L^2}.
\end{aligned}$$

Meanwhile, there exists a constant M depending only on Ω such that

$$\begin{aligned}\|\nabla_{\mathbf{u}^1} \delta \mathbf{d}\|_{L^2}^2 &\leq M(\|\operatorname{div}_{\mathbf{u}^1} \delta \mathbf{d}\|_{L^2}^2 + \|\operatorname{curl}_{\mathbf{u}^1} \delta \mathbf{d}\|_{L^2}^2) \\ &\leq C \left(\|\sqrt{t} \nabla \mathbf{d}^2\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau + \|\operatorname{curl}_{\mathbf{u}^1} \delta \mathbf{d}\|_{L^2}^2 \right),\end{aligned}$$

Observe that

$$\|\operatorname{div}_{\mathbf{u}^1} \delta \mathbf{d}\|_{L^2}^2 = \|(\operatorname{div}_{\mathbf{u}^2} - \operatorname{div}_{\mathbf{u}^1}) \mathbf{d}^2\|_{L^2}^2 \leq \|t^{-\frac{1}{2}} \delta A\|_{L^2}^2 \|\sqrt{t} \nabla \mathbf{d}^2\|_{L^\infty}^2 \leq C \|\sqrt{t} \nabla \mathbf{d}^2\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau.$$

Hence, choosing $\varepsilon > 0$ sufficiently small, we obtain that

$$\begin{aligned}&\frac{d}{dt} \left(\left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^2 + \|\delta \mathbf{d}\|_{L^2}^2 \right) + c_0 \left(\|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + \|\nabla \delta \mathbf{d}\|_{L^2}^2 \right) \\ &\leq C \left(\|\sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty}^2 + \|\sqrt{t} \nabla \mathbf{u}^2\|_{L^\infty}^2 + \|\sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty}^2 + \|\sqrt{t} \nabla \mathbf{d}^2\|_{L^\infty}^2 + \|\sqrt{t} \nabla Q^2\|_{L^4}^2 \right. \\ &\quad \left. + f_1(t) \right) \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau + C \|\tilde{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} \|\nabla \tilde{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^{\frac{1}{2}} + C \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2 + C e^{-2\beta_1 t} \|\delta \mathbf{d}\|_{L^2}^2 \\ &\quad + C \|\delta \rho_0\|_X^2 \left\| \sqrt{\rho_0^2} \mathbf{u}_t^2 \right\|_{L^2}^2\end{aligned}\tag{79}$$

for some $c_0 > 0$. Multiplying (79) by $e^{2\beta t}$, and adding $e^{2\beta t} \|\nabla \tilde{\mathbf{z}}\|_{L^2}^2$ to the both sides of (79), it follows from (59) that

$$\begin{aligned}&\frac{d}{dt} \left(\left\| e^{\beta t} \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^2 + \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 \right) + c_0 \left(\|e^{\beta t} \nabla \tilde{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \nabla \delta \mathbf{d}\|_{L^2}^2 \right) + \|e^{\beta t} \nabla \tilde{\mathbf{z}}\|_{L^2}^2 \\ &\leq 2\beta \left(\left\| e^{\beta t} \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^2 + \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 \right) + C \left(\|e^{\beta t} \sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla \mathbf{u}^2\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla Q^2\|_{L^4}^2 \right. \\ &\quad \left. + f_1(t) + \|e^{\beta t} \sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla \mathbf{d}^2\|_{L^\infty}^2 \right) \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau + C e^{2\beta t} \|\tilde{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} \|\nabla \tilde{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^{\frac{1}{2}} \\ &\quad + C e^{-2\beta_2 t} \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 + C e^{2\beta t} \left\| \sqrt{\rho_0^2} \mathbf{u}_t^2 \right\|_{L^2}^2 \|\delta \rho_0\|_X^2.\end{aligned}$$

In view of $\tilde{\mathbf{z}}|_{\partial\Omega} = 0$ and $\delta \mathbf{d} \cdot \mathbf{n}|_{\partial\Omega} = 0$, we obtain from (11) that

$$2\beta \left(\left\| e^{\beta t} \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^2 + \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 \right) \leq C\beta \left(\|e^{\beta t} \nabla \tilde{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \nabla \delta \mathbf{d}\|_{L^2}^2 \right).$$

Thus, by choosing β sufficiently small so that $C\beta \leq \frac{c_0}{4}$, we have

$$\begin{aligned}&\frac{d}{dt} \left(\left\| e^{\beta t} \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^2 + \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 \right) + \|e^{\beta t} \nabla \tilde{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \nabla \delta \mathbf{d}\|_{L^2}^2 + \|e^{\beta t} \nabla \tilde{\mathbf{z}}\|_{L^2}^2 \\ &\leq C \left(\|e^{\beta t} \sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla \mathbf{u}^2\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla Q^2\|_{L^4}^2 + f_1(t) + \|e^{\beta t} \sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty}^2 \right. \\ &\quad \left. + \|e^{\beta t} \sqrt{t} \nabla \mathbf{d}^2\|_{L^\infty}^2 \right) \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau + C e^{2\beta t} \|\tilde{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} \|\nabla \tilde{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \left\| \sqrt{\rho_0^1} \tilde{\mathbf{z}} \right\|_{L^2}^{\frac{1}{2}} \\ &\quad + C e^{-2\beta_2 t} \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 + C e^{2\beta t} \left\| \sqrt{\rho_0^2} \mathbf{u}_t^2 \right\|_{L^2}^2 \|\delta \rho_0\|_X^2.\end{aligned}\tag{80}$$

The decay bounds in (54) give

$$\|e^{\beta t} \sqrt{t} \nabla \mathbf{u}^1\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla \mathbf{u}^2\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla \mathbf{d}^1\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla \mathbf{d}^2\|_{L^\infty}^2 + \|e^{\beta t} \sqrt{t} \nabla Q^2\|_{L^4}^2 \leq f_1(t),$$

while the estimate (60) further implies

$$\begin{aligned} & \|e^{\beta t} \widehat{\mathbf{z}}_t\|_{L^{\frac{3}{2}}} \|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \\ & \leq \left(f_{\frac{4}{3}}(t) \left(\int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} + f_4(t) \|\nabla \delta \mathbf{u}\|_{L^2} \right) \|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

Applying $\|\nabla \delta \mathbf{u}\|_{L^2} \leq C(\|\nabla \widehat{\mathbf{z}}\|_{L^2} + \|\nabla \widehat{\mathbf{z}}\|_{L^2})$, we obtain from Young's inequality that

$$\begin{aligned} f_{\frac{4}{3}}(t) \left(\int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} & \leq f_1(t) \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau + \varepsilon \|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^2 \\ & \quad + f_1(t) \|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^2, \\ f_4(t) \|\nabla \delta \mathbf{u}\|_{L^2} \|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} \|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^{\frac{1}{2}} & \leq \varepsilon (\|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^2) \\ & \quad + f_1(t) \|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^2, \end{aligned}$$

whose small gradient contributions can be absorbed into the left-hand side of (80). Substituting these estimates into (80), we arrive at

$$\begin{aligned} & \frac{d}{dt} \left(\|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 \right) + \|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \nabla \widehat{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \nabla \delta \mathbf{d}\|_{L^2}^2 \\ & \leq f_1(t) \left(\|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 + \int_0^t e^{2\beta s} (\|\nabla \widehat{\mathbf{z}}\|_{L^2}^2 + \|\nabla \widehat{\mathbf{z}}\|_{L^2}^2 + \|\nabla \delta \mathbf{d}\|_{L^2}^2) ds \right) \\ & \quad + f_1(t) \|\delta \rho_0\|_X^2. \end{aligned} \tag{81}$$

Denoting by

$$G(t) = \|e^{\beta t} \sqrt{\rho_0^1} \widehat{\mathbf{z}}\|_{L^2}^2 + \|e^{\beta t} \delta \mathbf{d}\|_{L^2}^2 + \int_0^t e^{2\beta s} (\|\nabla \widehat{\mathbf{z}}\|_{L^2}^2 + \|\nabla \widehat{\mathbf{z}}\|_{L^2}^2 + \|\nabla \delta \mathbf{d}\|_{L^2}^2) ds, \tag{82}$$

we find that (81) reduces to

$$\frac{d}{dt} G(t) \leq f_1(t) G(t) + f_1(t) \|\delta \rho_0\|_X^2. \tag{83}$$

Since $G(0) = \|\sqrt{\rho_0^1} \delta \mathbf{v}_0\|_{L^2}^2 + \|\delta \mathbf{H}_0\|_{L^2}^2$, we obtain from Gronwall's inequality that

$$G(t) \leq \left(\|\sqrt{\rho_0^1} \delta \mathbf{v}_0\|_{L^2}^2 + \|\delta \mathbf{H}_0\|_{L^2}^2 \right) e^{\int_0^t f_1(s) ds} + \|\delta \rho_0\|_X^2 \int_0^t e^{\int_s^t f_1(\tau) d\tau} f_1(s) ds. \tag{84}$$

Consequently, it follows from (82) that

$$e^{2\beta t} \|\widehat{\mathbf{z}}(t)\|_{L^2}^2 \leq f_{1,\infty}(t) \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau \leq f_{1,\infty}(t) G(t). \tag{85}$$

Combining (84) and (85) yields

$$\begin{aligned} & \sup_{t \in [0, \infty)} e^{2\beta t} \left(\|(\rho_0^1)^{1/2} \delta \mathbf{u}\|_{L^2}^2 + \|\delta \mathbf{d}\|_{L^2}^2 \right) + \mu \int_0^\infty e^{2\beta t} \|\nabla \delta \mathbf{u}\|_{L^2}^2 dt + \nu \int_0^\infty e^{2\beta t} \|\nabla \delta \mathbf{d}\|_{L^2}^2 dt \\ & \leq C \left(\|(\rho_0^1)^{1/2} \delta \mathbf{v}_0\|_{L^2}^2 + \|\delta \mathbf{H}_0\|_{L^2}^2 + \|\delta \rho_0\|_X^2 \right), \end{aligned} \quad (86)$$

where $0 < \beta \leq \beta_0$ depends on $\mu, \nu, \rho^*,$ and Ω .

We next investigate the vacuum case. Since $\text{supp } \delta \rho_0 \subset \text{supp } \rho_0^2$ fails when $\rho_0^2 = 0$, the estimate (86) cannot be applied directly. To overcome this difficulty, we introduce the Eulerian intermediate solution $(\rho^{3/2}, \mathbf{v}^{3/2}, \mathbf{H}^{3/2}, P^{3/2})$ with the initial data $(\frac{1}{2}(\rho_0^1 + \rho_0^2), \mathbf{v}_0^2, \mathbf{H}_0^2)$. Its Lagrangian representation is denoted by $(\eta^{3/2}, \mathbf{u}^{3/2}, \mathbf{d}^{3/2}, Q^{3/2})$. Recall that the density is time-independent in Lagrangian coordinates, so that $\eta^i(t, y) = \rho_0^i(y)$ for $i = 1, 2, 3/2$. Hence, we have the support inclusion $\text{supp } \delta \rho_0^{\text{I,II}} \subset \text{supp } \frac{1}{2}(\rho_0^1 + \rho_0^2)$.

We introduce $(\delta \rho^{\text{I}}, \delta \mathbf{u}^{\text{I}}, \delta \mathbf{d}^{\text{I}})$ and $(\delta \rho^{\text{II}}, \delta \mathbf{u}^{\text{II}}, \delta \mathbf{d}^{\text{II}})$ by setting

$$\begin{aligned} (\delta \rho^{\text{I}}, \delta \mathbf{u}^{\text{I}}, \delta \mathbf{d}^{\text{I}}) & \triangleq \left(\frac{1}{2}(\rho_0^1 + \rho_0^2) - \rho_0^1, \mathbf{u}^{\frac{3}{2}} - \mathbf{u}^1, \mathbf{d}^{\frac{3}{2}} - \mathbf{d}^1 \right), \\ (\delta \rho^{\text{II}}, \delta \mathbf{u}^{\text{II}}, \delta \mathbf{d}^{\text{II}}) & \triangleq \left(\frac{1}{2}(\rho_0^1 + \rho_0^2) - \rho_0^2, \mathbf{u}^{\frac{3}{2}} - \mathbf{u}^2, \mathbf{d}^{\frac{3}{2}} - \mathbf{d}^2 \right). \end{aligned}$$

It follows immediately that

$$\delta \rho^{\text{I}} = -\delta \rho^{\text{II}} = \frac{1}{2}(\rho_0^2 - \rho_0^1), \quad \text{and} \quad \delta \mathbf{u} = \delta \mathbf{u}^{\text{I}} - \delta \mathbf{u}^{\text{II}}, \quad \delta \mathbf{d} = \delta \mathbf{d}^{\text{I}} - \delta \mathbf{d}^{\text{II}}. \quad (87)$$

In particular, $\delta \mathbf{u}^{\text{I}}|_{t=0} = \delta \mathbf{v}_0$, $\delta \mathbf{d}^{\text{I}}|_{t=0} = \delta \mathbf{H}_0$, $\delta \mathbf{u}^{\text{II}}|_{t=0} = 0$, $\delta \mathbf{d}^{\text{II}}|_{t=0} = 0$.

Inspired by [17, Subsection 2.2], to control the X -norm, we define

$$\|\delta \rho_0^{\text{I}}\|_X \triangleq \left\| \frac{\delta \rho_0^{\text{I}}}{\sqrt{\rho_0^1 + \rho_0^2}} \right\|_{L^4} = \left\| \frac{\delta \rho_0^{\text{II}}}{\sqrt{\rho_0^1 + \rho_0^2}} \right\|_{L^4} \leq \|\rho_0^1 - \rho_0^2\|_{L^2}^{\frac{1}{2}}, \quad (88)$$

provided that

$$\left| \frac{\rho_0^1 - \rho_0^2}{\sqrt{\rho_0^1 + \rho_0^2}} \right|^2 = |\rho_0^1 - \rho_0^2| \left| \frac{\rho_0^1 - \rho_0^2}{\rho_0^1 + \rho_0^2} \right| \leq |\rho_0^1 - \rho_0^2|.$$

Setting $\delta Q^{\text{I}} \triangleq Q^{3/2} - Q^1$, for $(\delta \rho^{\text{I}}, \delta \mathbf{u}^{\text{I}}, \delta \mathbf{d}^{\text{I}}, \delta Q^{\text{I}})$, the system (55) takes the form

$$\begin{cases} \rho_0^1 \delta \mathbf{u}_t^{\text{I}} - \Delta_{\mathbf{u}^1} \delta \mathbf{u}^{\text{I}} + \nabla_{\mathbf{u}^1} \delta Q^{\text{I}} = (\Delta_{\mathbf{u}^{3/2}} - \Delta_{\mathbf{u}^1}) \mathbf{u}^{3/2} - (\nabla_{\mathbf{u}^{3/2}} - \nabla_{\mathbf{u}^1}) Q^{3/2} + \mathbf{d}^{3/2} \cdot \nabla_{\mathbf{u}^{3/2}} \delta \mathbf{d}^{\text{I}} \\ \quad + \mathbf{d}^{3/2} \cdot (\nabla_{\mathbf{u}^{3/2}} - \nabla_{\mathbf{u}^1}) \mathbf{d}^1 + \delta \mathbf{d}^{\text{I}} \cdot \nabla_{\mathbf{u}^1} \mathbf{d}^1 - \delta \rho^{\text{I}} \mathbf{u}_t^{3/2}, \\ \delta \mathbf{d}_t^{\text{I}} - \Delta_{\mathbf{u}^1} \delta \mathbf{d}^{\text{I}} = (\Delta_{\mathbf{u}^{3/2}} - \Delta_{\mathbf{u}^1}) \mathbf{d}^{3/2} + \delta \mathbf{d}^{\text{I}} \cdot \nabla_{\mathbf{u}^1} \mathbf{u}^1 + \mathbf{d}^{3/2} \cdot \nabla_{\mathbf{u}^{3/2}} \delta \mathbf{u}^{\text{I}} + \mathbf{d}^{3/2} (\nabla_{\mathbf{u}^{3/2}} - \nabla_{\mathbf{u}^1}) \mathbf{u}^1, \\ \text{div}_{\mathbf{u}^1} \delta \mathbf{u}^{\text{I}} = (\text{div}_{\mathbf{u}^1} - \text{div}_{\mathbf{u}^{3/2}}) \mathbf{u}^{3/2}, \quad \text{div}_{\mathbf{u}^1} \delta \mathbf{d}^{\text{I}} = (\text{div}_{\mathbf{u}^1} - \text{div}_{\mathbf{u}^{3/2}}) \mathbf{d}^{3/2}, \\ \delta \mathbf{u}^{\text{I}}|_{t=0} = \delta \mathbf{v}_0, \quad \delta \mathbf{d}^{\text{I}}|_{t=0} = \delta \mathbf{H}_0. \end{cases} \quad (89)$$

Similarly, defining $\delta Q^{\text{II}} = Q^{3/2} - Q^2$, and replacing the superscript 1 and I in (89) by 2 and II, we see that $(\delta \rho^{\text{II}}, \delta \mathbf{u}^{\text{II}}, \delta \mathbf{d}^{\text{II}}, \delta Q^{\text{II}})$ satisfies the system (89).

Recalling the decomposition used in the non-vacuum case, we write

$$\delta \mathbf{u}^{\text{I}} = \widehat{\mathbf{z}}^{\text{I}} + \widetilde{\mathbf{z}}^{\text{I}}, \quad \delta \mathbf{u}^{\text{II}} = \widehat{\mathbf{z}}^{\text{II}} + \widetilde{\mathbf{z}}^{\text{II}},$$

where $(\widehat{\mathbf{z}}^{\text{I,II}}, \widetilde{\mathbf{z}}^{\text{I,II}})$ satisfy the system (61). Moreover, $\widehat{\mathbf{z}}^{\text{I,II}}$ obeys the boundary-value problems given in (56) and the weighted estimates (58)–(60).

Next, repeating the argument in the non-vacuum case, we define G^{I} and G^{II} as

$$\begin{aligned} G^{\text{I}}(t) &= e^{2\beta t} \left(\left\| \sqrt{\rho_0^1} \widetilde{\mathbf{z}}^{\text{I}}(t) \right\|_{L^2}^2 + \|\delta \mathbf{d}^{\text{I}}(t)\|_{L^2}^2 \right) \\ &\quad + \int_0^t e^{2\beta s} \left(\|\nabla \widetilde{\mathbf{z}}^{\text{I}}(s)\|_{L^2}^2 + \|\nabla \widehat{\mathbf{z}}^{\text{I}}(s)\|_{L^2}^2 + \|\nabla \delta \mathbf{d}^{\text{I}}(s)\|_{L^2}^2 \right) ds, \\ G^{\text{II}}(t) &= e^{2\beta t} \left(\left\| \sqrt{\rho_0^2} \widetilde{\mathbf{z}}^{\text{II}}(t) \right\|_{L^2}^2 + \|\delta \mathbf{d}^{\text{II}}(t)\|_{L^2}^2 \right) \\ &\quad + \int_0^t e^{2\beta s} \left(\|\nabla \widetilde{\mathbf{z}}^{\text{II}}(s)\|_{L^2}^2 + \|\nabla \widehat{\mathbf{z}}^{\text{II}}(s)\|_{L^2}^2 + \|\nabla \delta \mathbf{d}^{\text{II}}(s)\|_{L^2}^2 \right) ds. \end{aligned}$$

By virtue of (87), we then derive

$$\begin{aligned} \frac{d}{dt} G^{\text{I}}(t) &\leq f^{\text{I}}(t) G^{\text{I}}(t) + f^{\text{I}}(t) \|\delta \rho_0^{\text{I}}\|_X^2, \\ \frac{d}{dt} G^{\text{II}}(t) &\leq f^{\text{II}}(t) G^{\text{II}}(t) + f^{\text{II}}(t) \|\delta \rho_0^{\text{II}}\|_X^2, \end{aligned}$$

where $f^{\text{I}}, f^{\text{II}} \in L^1(\mathbb{R}_+)$. Since $\delta \mathbf{u}^{\text{I}}|_{t=0} = 0$ and $\delta \mathbf{d}^{\text{II}}|_{t=0} = 0$, it follows from (87) and Gronwall's inequality that

$$\begin{aligned} G^{\text{I}}(t) &\leq \left(\left\| \sqrt{\rho_0^1} \delta \mathbf{v}_0 \right\|_{L^2}^2 + \|\delta \mathbf{H}_0\|_{L^2}^2 \right) e^{\int_0^t f^{\text{I}}(\tau) d\tau} + \|\delta \rho_0^{\text{I}}\|_X^2 \int_0^t e^{\int_s^t f^{\text{I}}(\tau) d\tau} f^{\text{I}}(s) ds, \\ G^{\text{II}}(t) &\leq \|\delta \rho_0^{\text{I}}\|_X^2 \int_0^t e^{\int_s^t f^{\text{II}}(\tau) d\tau} f^{\text{II}}(s) ds. \end{aligned}$$

Summing the above inequalities and combining the resulting estimate with (88), we obtain that

$$\begin{aligned} &\sup_{t \in \mathbb{R}_+} e^{\beta t} \left(\|(\rho_0^1)^{1/2} \widetilde{\mathbf{z}}^{\text{I}}(t)\|_{L^2} + \|\delta \mathbf{d}^{\text{I}}(t)\|_{L^2} + \|(\rho_0^2)^{1/2} \widetilde{\mathbf{z}}^{\text{II}}(t)\|_{L^2} + \|\delta \mathbf{d}^{\text{II}}(t)\|_{L^2} \right) \\ &\quad + \|e^{\beta t} \nabla \delta \mathbf{u}\|_{L^2(\mathbb{R}_+; L^2)} + \|e^{\beta t} \nabla \delta \mathbf{d}\|_{L^2(\mathbb{R}_+; L^2)} \\ &\leq C \left(\|(\rho_0^1)^{1/2} \delta \mathbf{v}_0\|_{L^2} + \|\delta \mathbf{H}_0\|_{L^2} + \|\delta \rho_0\|_{L^2}^{\frac{1}{2}} \right). \end{aligned} \tag{90}$$

Finally, applying (85) to $(\widehat{\mathbf{z}}^{\text{I}}, \delta \mathbf{u}^{\text{I}}, G^{\text{I}})$ and $(\widehat{\mathbf{z}}^{\text{II}}, \delta \mathbf{u}^{\text{II}}, G^{\text{II}})$, and combining the resulting bounds with (90), we establish the stability estimate (9) in Theorem 1. $\square$

3 Proof of Theorem 2

This section is devoted to showing the stability in the Eulerian coordinates. We introduce negative Sobolev spaces to derive the estimates for the density, while the velocity and the

magnetic field are still considered within the L^2 framework. Following the arguments of [17, pp. 916–918], we have

$$\left\| \frac{\partial}{\partial s} X^s(t, \cdot) \right\|_{L^p(\Omega)} \leq C \int_0^t \|\delta \mathbf{v}\|_{L^p} dt', \quad \forall 1 \leq p < \infty,$$

$$\|\delta \rho(t, \cdot)\|_{W^{-1,p}} \leq C \left(\|\rho_0^2\|_{L^\infty} \left(\int_0^\infty e^{2\beta t} \|\delta \mathbf{v}\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \|\delta \rho_0\|_{L^2} \right),$$

where $(X^s)_{1 \leq s \leq 2}$ denotes the measure-preserving flow between X^1 and X^2 (see, e.g., [20]).

Furthermore, the estimate for $\nabla \delta \mathbf{v}$ follows from [17, pp. 918], and the estimate for $\nabla \delta \mathbf{H}$ can be obtained similarly. More precisely,

$$\|\nabla \delta \mathbf{v}\|_{L^2}^2 \leq C \left(\|\nabla \delta \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{v}^2\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau + \|\sqrt{t} \nabla^2 \mathbf{v}^1\|_{L^3}^2 \int_0^t \|\nabla \delta \mathbf{v}\|_{L^2}^2 d\tau \right), \quad (91)$$

$$\|\nabla \delta \mathbf{H}\|_{L^2}^2 \leq C \left(\|\nabla \delta \mathbf{d}\|_{L^2}^2 + \|\nabla \mathbf{H}^2\|_{L^\infty}^2 \int_0^t \|\nabla \delta \mathbf{u}\|_{L^2}^2 d\tau + \|\sqrt{t} \nabla^2 \mathbf{H}^1\|_{L^3}^2 \int_0^t \|\nabla \delta \mathbf{v}\|_{L^2}^2 d\tau \right). \quad (92)$$

After multiplying (91) by $e^{2\beta t}$ and integrating over $(0, t)$, Theorem 1, (54) (if $\beta \leq \beta_0$), and Gronwall's inequality imply

$$\int_0^t e^{2\beta s} \|\nabla \delta \mathbf{v}\|_{L^2}^2 ds \leq C \left(\left\| \sqrt{\rho_0^1} \delta \mathbf{v}_0 \right\|_{L^2}^2 + \|\delta \mathbf{H}_0\|_{L^2}^2 + \|\delta \rho_0\|_{L^2}^2 \right). \quad (93)$$

Then we deduce from (54), (92), (93), and Theorem 1 that

$$\int_0^t e^{2\beta s} \|\nabla \delta \mathbf{H}\|_{L^2}^2 ds \leq C \left(\left\| \sqrt{\rho_0^1} \delta \mathbf{v}_0 \right\|_{L^2}^2 + \|\delta \mathbf{H}_0\|_{L^2}^2 + \|\delta \rho_0\|_{L^2}^2 \right).$$

It remains to estimate the L^2 -norms of $\delta \mathbf{H}$ and $\sqrt{\rho^1} \delta \mathbf{v}$. Since the preceding argument is identical to that in [17, pp. 919], we only record the modified final estimates. For $\delta \mathbf{H}$, using the relation between the Eulerian and Lagrangian variables and the estimate for the difference of the flow maps, we find that

$$e^{\beta t} \|\delta \mathbf{H}\|_{L^2} \leq e^{\beta t} \|\delta \mathbf{d}\|_{L^2} + C \sqrt{t} e^{\beta t} \|\nabla \mathbf{H}^2\|_{L^6} \left(\int_0^t \|\nabla \delta \mathbf{v}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}}.$$

The Sobolev embedding, the elliptic regularity estimate, Lemma 4, (16), and (54) imply

$$\begin{aligned} \sup_{t \in \mathbb{R}_+} (\sqrt{t} e^{\beta t} \|\nabla \mathbf{H}^2\|_{H^1}) &\leq C + \sup_{t \in \mathbb{R}_+} (\|\nabla \mathbf{v}^2\|_{L^2} \|\sqrt{t} e^{\beta t} \mathbf{H}^2\|_{L^\infty}) + \sup_{t \in \mathbb{R}_+} (\|\nabla \mathbf{H}^2\|_{L^2} \|\sqrt{t} e^{\beta t} \mathbf{v}^2\|_{L^\infty}) \\ &\leq C. \end{aligned}$$

Hence, Theorem 1 and (93) give

$$\sup_{t \in \mathbb{R}_+} (e^{\beta t} \|\delta \mathbf{H}\|_{L^2}) \leq C \left(\left\| \sqrt{\rho_0^1} \delta \mathbf{v}_0 \right\|_{L^2} + \|\delta \mathbf{H}_0\|_{L^2} + \|\delta \rho_0\|_{L^2}^{\frac{1}{2}} \right).$$

For $\sqrt{\rho^1}\delta\mathbf{v}$, we similarly have

$$\|\sqrt{\rho^1}\delta\mathbf{v}\|_{L^2} \leq \left\| \sqrt{\rho_0^1}\delta\mathbf{u} \right\|_{L^2} + C\sqrt{t\rho^*}\|\nabla\mathbf{v}^2\|_{L^6} \left(\int_0^t \|\nabla\delta\mathbf{v}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}}. \quad (94)$$

The regularity properties of the Stokes system together with Lemma 3, (54), and Lemma 4 yield

$$\begin{aligned} \sup_{t \in \mathbb{R}_+} (\sqrt{t}e^{\beta t}\|\nabla\mathbf{v}^2\|_{L^6}) &\leq C + C \sup_{t \in \mathbb{R}_+} (\sqrt{\rho^*}\|\sqrt{t}e^{\beta t}\mathbf{v}^2\|_{L^\infty}\|\nabla\mathbf{v}^2\|_{L^2}) \\ &\quad + C \sup_{t \in \mathbb{R}_+} (\|\sqrt{t}e^{\beta t}\mathbf{H}^2\|_{L^\infty}\|\nabla\mathbf{H}^2\|_{L^2}) + C \sup_{t \in \mathbb{R}_+} (\|\sqrt{t}e^{\beta t}\mathbf{H}^2\|_{L^\infty}\|\nabla\mathbf{v}^2\|_{L^2}) \\ &\leq C. \end{aligned}$$

Combining this bound with Theorem 1, (93), and (94), we conclude that

$$\sup_{t \in \mathbb{R}_+} (e^{\beta t}\|\sqrt{\rho^1}\delta\mathbf{v}\|_{L^2}) \leq C \left(\left\| \sqrt{\rho_0^1}\delta\mathbf{v}_0 \right\|_{L^2} + \|\delta\mathbf{H}_0\|_{L^2} + \|\delta\rho_0\|_{L^2}^{\frac{1}{2}} \right).$$

The proof of Theorem 2 is complete. $\square$

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

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