

Simple, small knots in compression bodies

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Abstract: A knot in a 3-manifold is said to be small if the complement of the knot contains no essential incompressible surfaces. A topic in 3-manifold topology is that if one can tell the existence of small knots in a 3-manifold. The Lopez Conjecture guessed that every closed small 3-manifold contains small knots. R. Qiu and S. Wang constructed a class of simple, small knots in handlebodies. As an extension, the author proved that some compression bodies contain simple, small knots. That is: let F be a connected closed surface with genus at least 2. Let C be a compression body obtained by attaching $2g(F)-1$ 1-handles to $F \times I$, then there are simple, small knots in C .

Keywords: Simple knots; Small knots; incompressible surfaces

15 0 Introduction

In this paper, all manifolds are supposed to be piecewise linear and orientable and all submanifolds are supposed to be properly imbedded. Let F be a surface, the genus of F denoted by $g(F)$. Let I be the closed interval $[0,1]$.

Let F be a properly embedded surface in a 3-manifold M . F is said to be compressible if F is a 2-sphere which bounds a 3-ball, or there is a disk $D \subset M$ such that $D \cap F = \partial D$ is an essential curve in F ; otherwise, F is said to be incompressible. F is said to be ∂ -compressible if there is a disk D with $\partial D = u \cup v$ such that $D \cap F = u$ is an essential arc and $D \cap \partial M = v$; otherwise, F is said to be ∂ -incompressible. An incompressible, ∂ -incompressible surface in M is essential if it is not parallel to ∂M .

25 Suppose M is a 3-manifold, M is said to be

- (1) irreducible, if M contains no essential 2-spheres;
- (2) ∂ -irreducible, if ∂M is incompressible;
- (3) anannular, if it contains no essential annuli;
- (4) atoroidal, if it contains no essential tori.

30 A 3-manifold is said to be simple, if it is irreducible, ∂ -irreducible, anannular and atoroidal. A 3-manifold is said to be small, if it contains no essential closed surfaces. A knot k in a 3-manifold is said to be small (resp. simple) if the complement of k is small (resp. simple).

Note that a simple 3-manifold admits a complete finite volume hyperbolic structure with totally geodesic boundary (with torus components removed, See [1]).

35 Suppose F is a closed surface, the connected manifold C obtained by attaching some 1-handles to $F \times I$ along $F \times \{1\}$ is called a compression body. $F \times \{0\}$ is called the negative boundary of C , denoted by $\partial_- C$, $\partial C - \partial_- C$ is called the positive boundary of C , denoted by $\partial_+ C$.

40 A topic in 3-manifold topology is that if one can tell the existence of small knots in a 3-manifold. The Lopez Conjecture guessed that every closed small 3-manifold contains small

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knots, see [2]. In [3], H. Matsuda constructed a class of small knots in some Haken manifolds. In [4] and [5], R. Qiu and S. Wang constructed a class of simple, small knots in handlebodies. As an extension of [4], we proved that some compression bodies contain simple, small knots.

Theorem 1. Suppose F is a connected closed surface of genus at least two, and C is a compression body obtained by attaching $2g(F)-1$ 1-handles to $F \times I$, then there are simple, small knots in C .

1 Lemmas on incompressible surfaces in 3 manifolds

The following lemmas are basic lemmas in studying combinatorial topology of 3-manifolds, and they are very useful in our proofs.

Lemma 1.1. Suppose F is an incompressible surface in an irreducible 3-manifold M . If there is a proper embedded disk D in M such that $\partial D \subset \partial F$, then F is a disk which is isotopic to D .

Lemma 1.2. Suppose F is an incompressible surface in an irreducible 3-manifold M , and A is an essential annular in M with $\partial A \subset \partial F$. If there is a disk D in M such that $D \cap (F \cup A) = \partial D = u \cup v$, where u is an essential arc on F and v is an essential arc on A , then F is an annular which is isotopic to A .

Lemma 1.3. Let F be a closed surface, and S be an incompressible surface in $F \times I$ and $\partial S \subset F \times \{1\}$ (resp. $\partial S \subset F \times \{0\}$), then S is parallel to $F \times \{1\}$ (resp. $F \times \{0\}$).

Lemma 1.4. Let F be a surface in an irreducible 3-manifold M , and $M' = M - \text{int } N(F)$. Let S be an incompressible surface in M which intersects F and $|S \cap F|$ is minimal, then $S \cap M'$ is incompressible in M' .

2 Construction of the simple, small knots

Suppose F is a connected closed surface with genus $g(F)$ at least two. Let C be a compression body obtained by attaching n 1-handles to $F \times I$ along $F \times \{1\}$ where $n = 2g(F)-1$. Since $g(F) \geq 2$, then $n \geq 3$.

The construction of the simple, small knots in C is obtained by pushing a circle l on $\partial_+ C$ into $\text{int } C$, where l is a closed curve with one self intersection.

By the definition of C , there are n essential separating disks E_1, E_2, \dots, E_n in C such that $C = M_0 \cup_{E_1} J_1 \cup_{E_2} J_2 \cup \dots \cup_{E_n} J_n$, where $M_0 = F \times I$ and each J_i is a solid torus. Let F be $F \times \{1\}$. See Fig. 1.

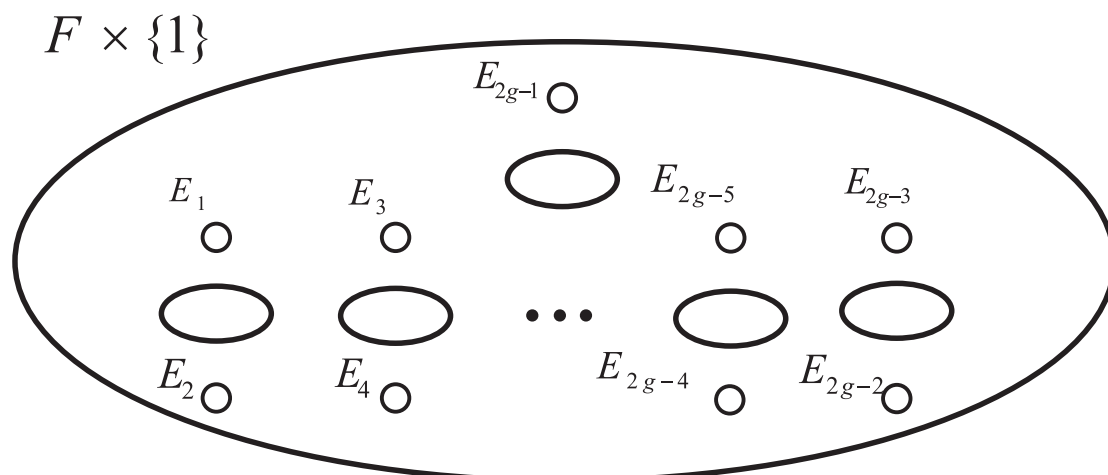


Fig. 1 $F = F \times \{1\}$

Choose $2n$ arcs a_i ($i = 1, 2, \dots, 2n$) on F as in Fig.2. Furthermore, the arcs are chosen to satisfy the following proposition.

Proposition 2.1. (1) $F - (\bigcup_{i=1}^n E_i) - (\bigcup_{i=1}^{2n} N(a_i))$ is a disk,

(2) there are exactly four endpoints of the arcs incident to each disk E_i for $i = 1, 2, \dots, n$,

(3) each arc of $\bigcup a_i$ has its two endpoints incident to different disks,

(4) by traveling along all the arcs under the way that from one arc to the opposite arc at each E_i , a Euler circuit will be obtained.

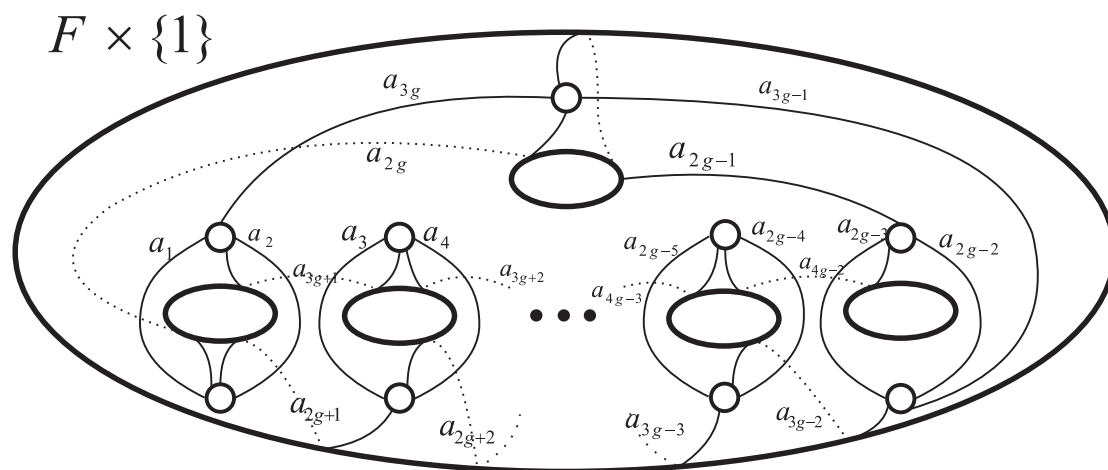


Fig. 2 curves on F

Fig 2 shows a specific example of the arcs satisfying Proposition 2.1.

In each of J_i for $i = 1, 2, \dots, n-1$, we choose two arcs b_{2i-1} and b_{2i} as in Fig. 3(a). Specially, choose b_{2n-1} and b_{2n} as in Figure 3(b). Note that b_{2n-1} intersects b_{2n} in one point.

Furthermore, $\partial(\bigcup a_i) = \partial(\bigcup b_i)$.

Let $l = (\bigcup a_i) \cup (\bigcup b_i)$, by Proposition of 2.1 (4), l is a closed curve on $\partial_+ C$ with one self intersection point.

Note. The arcs in J_i are chosen under the same way as R. Qiu and S. Wang's, for more details, see [4] and [5]

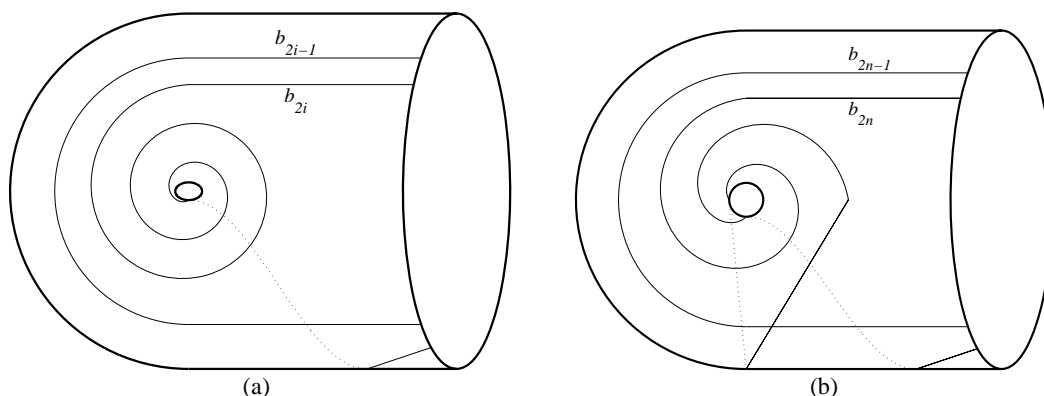


Fig. 3 curves on F

Now let k be the knot obtained by pushing l into $\text{int } C$ in the following way: Firstly push b_{2n} deeply, and then push $l - b_{2n}$ slightly into $\text{int } C$ so that b_{2n} is under crossing b_{2n-1} . We denote by a'_i and b'_j the arcs obtained by pushed from a_i and b_j respectively. Now $k = (\bigcup_{i=1}^{2n} a'_i) \cup (\bigcup_{j=1}^{2n} b'_j)$.

Let $N(k)$ be the regular neighborhood of k in C . Let $C_k = C - \text{int } N(k)$, $N_0 = M_0 - \text{int } N(k)$, $N_i = J_i - \text{int } N(k)$ and $F_i = E_i - \text{int } N(k)$ for $i = 1, 2, \dots, n$. Let $T = \partial N(k)$. Let $X = \{a_i \mid i = 1, 2, \dots, 2n\} \cup \{b_j \mid j = 1, 2, \dots, 2n\}$. Note that T is divided by $\bigcup E_i$ into $4n$ annuli $\bigcup_{x \in X} A_x$, where $A_x = (\partial N(x')) \cap T$. By the construction, k and l bound a non-embedded annulus A_* in C (the trace of pushing l to k) which is cut by $\bigcup E_i$ into $4n$ disks $\bigcup D_{a_i}$ and $\bigcup D_{b_j}$. Note that $D^* = \bigcup_{x \in X - \{b_{2n}\}} D_x$ is still a disk.

Lemma 2.2. (1) k is not contractible in C .

(2) There is no relative homotopy on (J_i, E_i) which either send b_{2i-1} to b_{2i} , or send b_{2i-1} (respectively b_{2i}) to E_i .

(3) (N_n, F_n) is homeomorphism to (N_i, F_i) for each $i = 1, 2, \dots, n-1$.

(4) For $i = 0, 1, \dots, n$, let A be an essential annulus in N_i such that $\partial_1 A \subset \bigcup F_j$ and $\partial_2 A \subset A_x$ for $x \in X$, then $\partial_1 A$ is parallel to ∂A_x on one component of $\bigcup F_j$, say F_{j^*} and A can be isotopic to F_{j^*} by fixing $\partial_1 A$.

Proof. (1), (2) and (3) are held by a directly observation of the constructions.

For (4), by (3) we can suppose that $\partial_1 A \subset F_1$ and A is contained in N_0 or N_1 . Let D be the disk bounded by $\partial_1 A$ from F_1 .

In the case that A is in N_1 , $\partial_1 A$ bounds a disk E in J_1 which intersects b'_1 only in one point and is disjoint from b'_2 . Since J_1 is a solid torus, E cuts off a 3-ball B^3 from J_1 . If one component of $\partial b'_2$ is contained in D , then b'_2 is contained in B^3 . This contradicts (2).

It is also impossible that both the two endpoints of $\partial b'_1$ is contained in D . Hence $\partial_1 A$ is parallel to a component c of $F_1 \cap A_{b_1}$. Then $\partial_1 A$ and c bounds an annulus A_1^* from F_1 , and c and $\partial_2 A$ bounds an annulus A_2^* from A_{b_1} . Suppose $|A \cap D_{b_1}|$ is minimal. Then
 125 $A \cap D_{b_1}$ has only one arc, and it cuts off a disk from D_{b_1} . By Lemma 1.2, A is parallel to $A_1^* \cup A_2^*$. Hence (4) holds.

In the case that A is in N_0 , $\partial_1 A$ bounds a disk E in M_0 which intersects $\bigcup a'_i$ within only one point. Since M_0 is irreducible, E cuts off a 3-ball B^3 from M_0 . By Proposition 2.1 (3), B^3 can only meet one arc of $\bigcup a'_i$. That means that D can contain only
 130 one component of $F_1 \cap T$. By the same argument as previous case, (4) holds. \square

Propositon 2.3. C_k is irreducible.

Proof. This directly follows from Lemma 2.2 (1). \square

3 C_k is ∂ -irreducible

Lemma 3.1. $\bigcup_{i=1}^n F_i$ is incompressible and ∂ -incompressible in N_i , for $i = 1, 2, \dots, n$.

Proof. See Lemma 3.2 of [4]

\square

Lemma 3.2. $\bigcup_{i=1}^n F_i$ is incompressible in N_0 .

Proof. Suppose D is a compressible disk of F_i . Let B be a sub-disk of E_i which is bounded by ∂D . After a slightly push, $B \cup D$ is a separating 2-sphere in $\text{int } M_0$ and
 140 $B \cup D$ intersects some a_j within exactly one point. Since each a_j is parallel to the boundary of M_0 , it is a contradiction. \square

Lemma 3.3. $\bigcup_{i=1}^n F_i$ is ∂ -incompressible in N_0 .

Proof. Suppose D is a ∂ -compressible disk of F_i , and $\partial D = u \cup v$, where u is an essentail arc on $\bigcup_{i=1}^n F_i$ and v is an arc on ∂N_0 . By Proposition 2.1 (3),
 145 $v \subset \partial_+ M_0 - \bigcup_{i=1}^n \partial E_i$. Note that u separates some E_i into two disks E_i^1 and E_i^2 . Since E_i is incompressible, v is essential on $\partial_+ M_0 - \bigcup_{i=1}^n \partial E_i$. Hence both $E_i^1 \cup D$ and $E_i^2 \cup D$ are ∂ -reducing disks of $\partial_+ M_0$. This is a contradiction for M_0 is ∂ -irreducible. \square

Proposition 3.4. C_k is ∂ -irreducible.

Proof. Suppose C_k is ∂ -reducible with E is a ∂ -reducible disk of ∂C_k . Since $\partial_- C$
 150 is incompressible in C , ∂E can not lie in $\partial_- C$.

Suppose $\partial E \subset T$, then either k is contractible or C_k contains an essential 2-sphere. In either case it is a contradiction.

Now we suppose $\partial E \subset \partial_+ C$. Furthermore, we assume that $|E \cap (\bigcup F_i)|$ is minimal

among all ∂ -reducing disks of C_k .

155 Firstly, we suppose that $E \cap (\bigcup F_i) \neq \emptyset$. An innermost circle component of $E \cap (\bigcup F_i)$ on E indicates that $\bigcup F_i$ is compressible; And an outermost arc component of $E \cap (\bigcup F_i)$ on E will indicates that $\bigcup F_i$ is either compressible or ∂ -compressible. In each case it is a contradiction since $\bigcup F_i$ is incompressible and ∂ -incompressible.

Now we suppose $E \cap (\bigcup F_i) = \emptyset$. By the argument in the proof of case 1 of Lemma 3.3 in [4], can not be contained in N_i for $i=1,2,\dots,n$, then it must be contained in N_0 . Since 160 M_0 is ∂ -irreducible, ∂E is trivial on $\partial_+ M$. Then ∂E bounds a disk D on $\partial_+ M$ such that there are some component of $\bigcup E_i$ contained in $\text{int } D$. Moreover E separates M_0 into a 3-ball B_0 and M'_0 , where M'_0 is homeomorphism to M_0 . Since k is disjoint from E , $E_i \subset D$ for all $i=1,2,\dots,n$. Suppose E is chosen such that $|E \cap (\bigcup D_{a_i})|$ is minimal.

165 Then E must be disjoint from $\bigcup D_{a_i}$, otherwise, an outermost disk of $\bigcup D_{a_i}$ cutting by E will produce a new reducing disk E' of ∂N_0 with $|E' \cap (\bigcup D_{a_i})| < |E \cap (\bigcup D_{a_i})|$. Now all the disks $\bigcup E_i$ and arcs $\bigcup a_i$ are contained in D , it is a contradiction since $F - (\bigcup E_j) - (\bigcup a_i)$ is a disk by Proposition 2.1 (1). \square

4 C_k is anannular

170 **Proposition 4.1.** C_k is anannular.

Proof. Suppose otherwise that C_k contains an essential annulus A . Assume that $|A \cap (\bigcup F_i)|$ is minimal among all the essential annuli in C_k , and also assume $|A \cap (\bigcup D_{a_i})|$ is minimal among all such annuli. Since $\bigcup F_i$ is incompressible and ∂ -incompressible, all the components of $|A \cap (\bigcup F_i)|$ is essential on both A and $\bigcup F_i$.

175 Let $\partial A = \partial_1 A \cup \partial_2 A$, and let μ be the meridian sloop of T . There are six cases.

Case 1. $\partial A \subset T$. There are two subcases:

Subcase 1.1. $\Delta(\partial_1 A, \mu) = 0$. We may assume that $\partial A \cap (\bigcup F_i) = \emptyset$. Then $A \cap (\bigcup F_i)$ consists of essential circles in both A and $\bigcup F_i$.

180 Firstly, we assume that $A \cap (\bigcup F_i) \neq \emptyset$. Let A_i^* be the outermost annulus cut off by $\bigcup F_i$ from A . That is there is a circle c of $A \cap F_{i^*}$ for some $i^* \in \{1,2,\dots,n\}$ such that $\partial A_1^* = \partial_1 A \cup c$ and $\text{int } A_1^*$ is disjoint from $\bigcup F_i$. By Lemma 2.2 (4), A_1^* is parallel to F_{i^*} . Then A_1^* can be pushed to be disjoint from F_{i^*} (if there are other components of $A \cap F_{i^*}$, then push them together). This will reduce $|A \cap (\bigcup F_i)|$. Hence it is a contradiction.

Now we assume that $A \cap (\bigcup F_i) = \emptyset$, then A is contained in some N_i , where

185 $i = 0, 1, \dots, n$. By Lemma 2.2 (3), we can suppose A is contained in either N_0 or N_1 . Recall the disk D_x and the annulus A_x , where $x \in \{b_1, b_2\} \cup \{a_i \mid i = 1, 2, \dots, n\}$. Since A is incompressible, then we can suppose that $A \cap D_x$ contains only arcs, and $\partial_1 A$ and $\partial_2 A$ are contained in the same annulus A_{x^*} for some $x^* \in \{b_1, b_2\} \cup \{a_i \mid i = 1, 2, \dots, n\}$. Let A^1 be the annulus bounded by $\partial_1 A$ and $\partial_2 A$ from A_{x^*} . The arc of $A \cap D_{x^*}$ cuts off a disk from
190 D_{x^*} which servers Lemma 1.2 see Fig4. Hence A is parallel to $A^1 \subset A_{x^*}$. That is A is parallel to T . It is a contradiction.

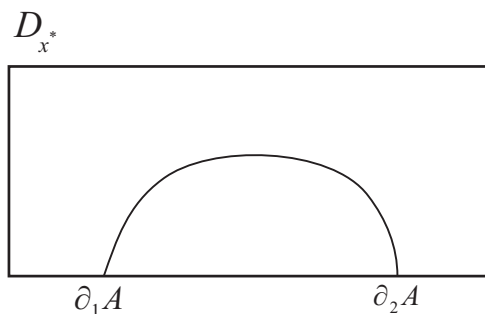


Fig. 4 curves on F

195 **Subcase 1.2.** $\Delta(\partial_1 A, \mu) \geq 1$. Now, $A \cap F_1$ consists of essential arcs in A , which cut A into $4\Delta(\partial_1 A, \mu)$ rectangles. Consider the rectangles contained in N_1 . Each rectangle has two opposite sides on F_1 and two opposite sides on A_x and A_y , where $x, y = b_1$ or b_2 . If $x \neq y$, it contradicts Lemma 2.2 (2). Hence $x = y$. Then the two endpoints of each arc of $A \cap F_1$ lie in a same component of ∂F_1 . Furthermore each component of $\partial F_1 \cap T$ has
200 $\Delta(\partial_1 A, \mu)$ endpoints of the arcs of $A \cap F_1$. An innermost arc is trivial on F_1 and cuts off a disk from F_1 which servers Lemma 12. Hence A is parallel to T and it is a contradiction.

Case 2. $\partial_1 A \subset T$ and $\partial_2 A \subset \partial_+ C$. By Proposition 3.4, C_k is ∂ -irreducible. Clearly C_k is not homeomorphic to $T \times I$. Since both Dehn fillings along μ and $\partial_1 A$ compress $\partial_+ C$, by Lemma 2.4.3 in [6], $\Delta(\partial_1 A, \mu) \leq 1$. There are two subcases.

205 **Subcase 2.1.** $\Delta(\partial_1 A, \mu) = 0$. Suppose $\partial_1 A$ is disjoint from $\bigcup F_i$. Then $\partial_2 A$ is disjoint from $\bigcup F_i$, otherwise each arc of $A \cap (\bigcup F_i)$ is trivial in A .

Firstly, suppose that $A \cap (\bigcup F_i) \neq \emptyset$. Let c be an innermost circle of $A \cap (\bigcup F_i)$ on A , that is $\partial_1 A$ and c bounds a sub-annulus A_1^* of A such that $\text{int } A_1^*$ is disjoint from $\bigcup F_i$. By the same argument in proving subcase 1.1, A_1^* can be isotopic to be disjoint from $\bigcup F_i$ and
210 then $|A \cap (\bigcup F_i)|$ is reduced.

Now, suppose that A is disjoint from $\bigcup F_i$. By the proof of Case 2 (1) of Lemma 3.4 in [4], A can not be contained in N_i for $i = 1, 2, \dots, n$. Suppose A is contained in N_0 . Let D be a meridian disk of $N(k)$ with $\partial_1 A = \partial D$. Let $B = A \cup_{\partial_1 A} D$. Then B is a proper

disk in M_0 and ∂B is non-trivial in $F - (\bigcup E_i)$ but trivial in F . Suppose that A is isotopic to the situation such that $|A \cap (\bigcup_{x \in X} D_x)|$ is minimal. Then A is disjoint from $\bigcup_{x \neq a_1} D_x$. Hence $\partial_2 A$ is disjoint from $\bigcup_{i \neq 1} a_i$. Then ∂B contains all the disks of $\bigcup E_i$ and all the arcs of $\bigcup_{i \neq 1} a_i$ by Proposition 2.1 (4). By Proposition 2.1 (1), it is impossible.

Subcase 2.2. $\Delta(\partial_1 A, \mu) = 1$. This case is ruled out by the same argument as in the proof of Case 2 (2) of Lemma 3.4 in [4].

Case 3. $\partial_1 A \subset T$ and $\partial_2 A \subset \partial_- C$. By the argument in proving Case 2, there are two subcases.

Subcase 3.1. $\Delta(\partial_1 A, \mu) = 0$. In this case, we can find a reducing disk of $\partial_- C$ in C . It is a contradiction.

Subcase 3.2. $\Delta(\partial_1 A, \mu) = 1$. In this case, $k \sim 0$ in $H(C, \partial_- C)$. It is obviously a contradiction.

Case 4. $\partial A \subset \partial_+ C$.

Firstly, suppose that $A \cap (\bigcup F_i) = \emptyset$. Then A is contained in one of N_0, N_1, \dots, N_n . By the argument in the proof of Case 3 of Lemma 3.4 in [4], A cannot be contained in N_1, N_2, \dots, N_n . Hence A is contained in N_0 . Since A is essential, A is disjoint from D_{a_i} . This means that ∂A is contained in the disk $F - (\bigcup E_j) - (\bigcup a_i)$ by Proposition 2.1 (1) and (4). It is a contradiction.

Now suppose that $A \cap (\bigcup F_i) \neq \emptyset$. Then there are two subcases:

Subcase 4.1. Each component of $A \cap (\bigcup F_i)$ is an essential circle in A .

Suppose c is an outermost circle of $A \cap (\bigcup F_i)$ on A , that is c and $\partial_1 A$ bound a sub-annulus A_1^* of A so that $\text{int } A_1^*$ is disjoint from $\bigcup F_i$. Without loss of generality, assume that $c \subset (A \cap F_1)$. By the arguments in the proof of Case 3 (1) of Lemma 3.4 in [4], A_1^* cannot be contained in N_1 . Now A_1^* is contained in N_0 .

Let $a_{i_1}, a_{i_2}, a_{i_3}$ and a_{i_4} be the four arcs of $\bigcup a_i$ which incident to E_1 , see section 2. Then $\partial_1 A \cap a_i = \emptyset$, for $i \neq i_1, i_2, i_3$ or i_4 , and $\partial_1 A$ intersects a_i with only one point for $i = i_1, i_2, i_3$ and i_4 . Otherwise, do a 2-surgery along an outermost disk cut off from $\bigcup D_{a_i}$ by $A \cap (D_{a_i})$, will reduce $|A \cap (\bigcup D_{a_i})|$.

Claim 1. $\partial_1 A$ is parallel to ∂E_1 .

Proof. Note that c bounds a disk in E_1 , then $\partial_1 A$ bounds a disk in M_0 . Since M_0 is ∂ -irreducible, $\partial_1 A$ bounds a disk D on F and D includes some components of $\bigcup E_i$.

Suppose otherwise that $\partial_1 A$ is not parallel to ∂E_1 . This means that D contains E_{i^*} for some $i^* \neq 1$. By Proposition 2.1 (2), (3) and (4), $\text{int } D$ contains all the components of $\bigcup_{i=1}^n E_i$, but it doesn't contain E_1 . Furthermore $\text{int } D$ contains all the arcs of

$(\bigcup a_i - \{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\})$. By Proposition 2.1, each component of $F - (\bigcup_{i=1}^n E_i) - (\bigcup_{j=1}^{2n} a_j) - \partial_1 A$ is a disk. On F , by taking D and E_1 as two vertices, we obtain a graph with four edges and at least one disk faces. Hence $\chi(F) \geq 2 - 4 + 1 = -1$. It contradicts the fact that $g(F) \geq 2$. Hence $\partial_1 A$ is parallel to ∂E_1 . \square

By Claim 1, $\partial_1 A$ and ∂E_1 bound an annulus A^* on F .

Claim 2. On F_1 , c is parallel to ∂E_1 .

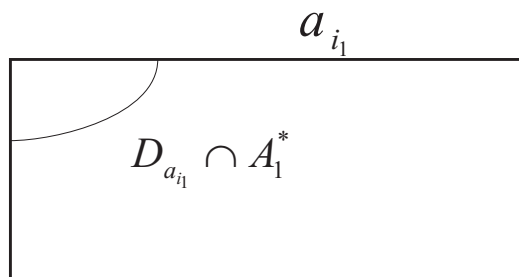


Fig. 5 A_1^* intersects $D_{a_{i_1}}$

Proof. Since $\partial_1 A$ intersects a_{i_1} in one point, $A_1^* \cap D_{a_{i_1}}$ is a single arc and furthermore c intersects $\partial D_{a_{i_1}} \cap F_1$ within a single point, see Fig. 5. By the same reason, c intersects each of $\partial D_{a_{i_2}} \cap F_1$, $\partial D_{a_{i_3}} \cap F_1$ and $\partial D_{a_{i_4}} \cap F_1$ within only one point. Hence c bounds a disk C^* on E_1 such that C^* contains the four components of $N(k) \cap E_1$. That is c is parallel to ∂E_1 . \square

By Claim 2, c and ∂E_1 bound an annulus B^* on F_1 . On $D_{a_{i_1}}$, the single arc (see Fig.5) of $A_1^* \cap D_{a_{i_1}}$ cuts off a ∂ -compressible disk of A_1^* to $A^* \cup B^*$. By Lemma 1.2, A_1^* is parallel to $A^* \cup B^*$. Hence we can isotopy A_1^* from N_0 to N_1 to reduce $|A \cap (\bigcup F_i)|$. It is a contradiction.

Subcase 4.1. Each component of $A \cap (\bigcup F_i)$ is an essential arc in A . Then $\bigcup F_i$ cuts A into rectangles R_j in N_i , each R_j has two opposite sides in $\bigcup F_i$ and remaining two sides on $\partial_+ C$.

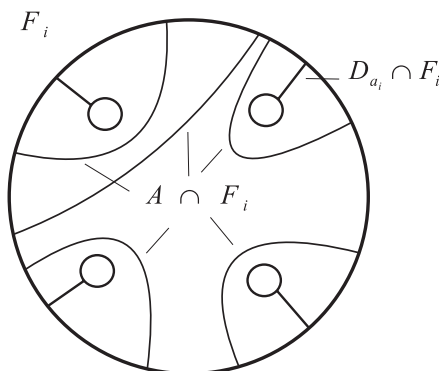


Fig. 6 $A \cap F_i$

For R_j in N_i for $i \geq 1$, by the proof of Lemma 3.4 Case 3 (2) in [4], R_j is essential in N_i and R_j cuts off a solid torus from N_i such that the solid torus is homeomorphism to $A_{b_{2i-1}} \times I$ or $A_{b_{2i}} \times I$. After a proper isotopy, we can assume that $A \cap F_i$ is disjoint from $(\bigcup D_{a_i}) \cap F_i$. See Fig. 6. Furthermore we can suppose that A is disjoint from $\bigcup D_{a_i}$ for
 275 $|A \cap (\bigcup D_{a_i})|$ is minimal.

For R_j in N_0 , since $F \times I$ is ∂ -irreducible, ∂R_j must bound a disk D_R on F . If D_R contains more than one arcs of $\bigcup a_i$, then D_R contains all the components of $\bigcup E_i$ and all the components of $\bigcup a_i$. It is a contradiction to Proposition 2.1 (1). Hence D_R contains only one arc of $\bigcup a_i$, say a_{i^*} . That means that R_j cuts off a solid torus from N_0 which is
 280 homeomorphism to $A_{a_{i^*}} \times I$.

When all the solid tori cutting by R_j from N_i for $0 \leq i \leq n$ are attached together, the resulting manifold is homeomorphism to a $torus \times I$ with one boundary is $\partial N(k)$. The span annulus of it is an essential annulus in C_k and has its two boundaries on T and $\partial_+ C$ respectively. This is a contradiction to Case 2.

285 **Case 5.** $\partial A \subset \partial_- C$. Since C contains no essential annuli with both of its boundaries on $\partial_- C$, A must be parallel to $\partial_- C$. That means A cuts off a solid torus J_A from C . Since A is essential in C_k , k is contained in J_A . Then, $k \sim 0$ in $H(C, \partial_- C)$. It is obviously a contradiction.

Case 6. $\partial_1 A \subset \partial_+ C$ and $\partial_2 A \subset \partial_- C$. Suppose A is such an annulus such that
 290 $|A \cap (\bigcup E_i)|$ and $|A \cap (\bigcup D_{a_i})|$ are minimal.

By a standard outermost disk argument, we can assume that A is disjoint from $\bigcup E_i$. Then A is contained in N_0 . Also by a standard outermost disk argument, A is disjoint from $\bigcup D_{a_i}$. Since $F - (\bigcup E_i) - (\bigcup a_i)$ is a disk, $\partial_1 A$ is trivial on $\partial_+ C$. It contradicts that A is essential.

295 5 C_k contains no closed essential surfaces

For $x \in X - \{b_{2n}\}$, let $W_x = \partial N(D_x \cup A_x) - \partial_+ C - \bigcup_{i=1}^n F_i$ and $W = \bigcup_{x \neq b_{2n}} W_x$. Note that F_i is a disk with four punctures. It has five boundaries, one is ∂E_i , the other four boundaries are $\partial N(k) \cap F_i$. Each component of $(\bigcup_{x \neq b_{2n}} D_x) \cap F_i$ is a single arc which connects ∂E_i to one component of $T \cap F_i$. $W \cap F_i$ are four arcs, each of which cuts off a
 300 disk containing one component of $T \cap F_i$ from E_i . See Fig. 8 and Fig. 9.

Suppose C_k contains a closed essential surface P . We define the complexity of P by an ordered pair $C(P) = (|P \cap W|, |P \cap (\bigcup F_i)|)$. Furthermore, we suppose P realizes the

minimality of $C(P)$. By a standard innermost argument, we have:

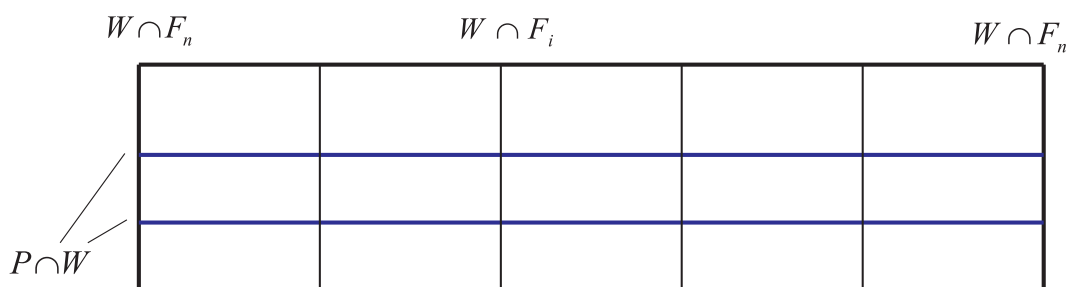


Fig. 7 $P \cap W$

Lemma 5.1.

(1) Each component of $P \cap (\bigcup F_i)$ is an essential circle on both P and $\bigcup F_i$.

(2) Each component of $P \cap W$ is an arc connecting the two components of $W \cap F_n$

(the two sides of the rectangle W). See Fig. 7.

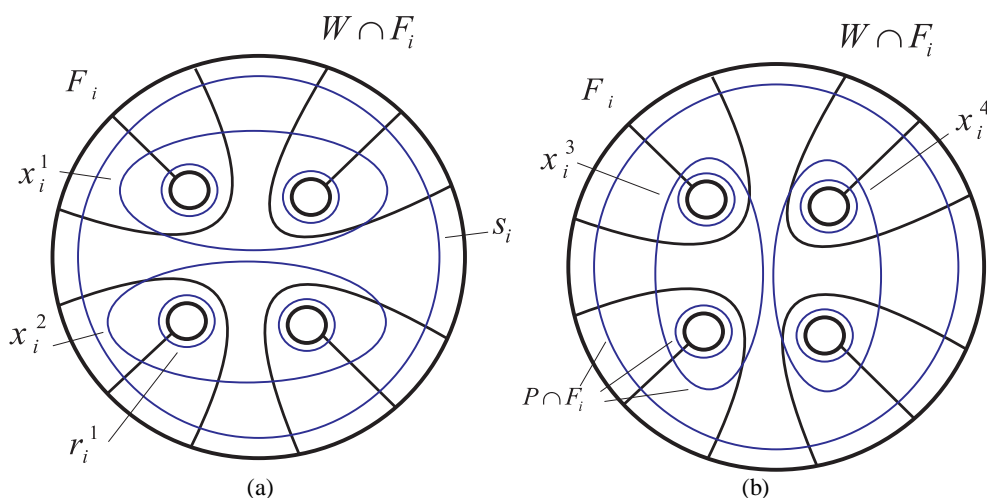


Fig. 8 $P \cap F_i$

Lemma 5.2. The pattern of $P \cap F_i$ for $i = 1, 2, \dots, n$ is as in Fig. 8. The components of $P \cap F_i$ can be divided into three types of isotopic classes:

s_i , which is isotopy to ∂E_i ;

x_1^i, x_2^i, x_3^i and x_4^i , each of which bounds a disk of E_i containing two components of $T \cap E_i$;

r_1^i, r_2^i, r_3^i and r_4^i , each of which is parallel to a component of $T \cap E_i$. See Fig. 8.

Proof. See the proof of Lemma 4.2 in [4]. \square

For each class μ in Lemma 5.2, denoted by $v(\mu)$ the number of components in μ .

Lemma 5.3. For $i = 1, 2, \dots, n-1$, each component of $P \cap N_i$ is isotopic to one of $N_i \cap \partial_+ C$, $A_{b_{2i}}$ or $A_{b_{2i-1}}$, and the pattern of $P \cap F_i$ is as in Fig. 9. Furthermore $v(s_i)$

(resp. $v(r_i^m)$ for $m = 1, 2, 3, 4$) are all equal for $i = 1, 2, \dots, n-1$.

Proof. See the proof of Lemma 4.2 in [4]. \square

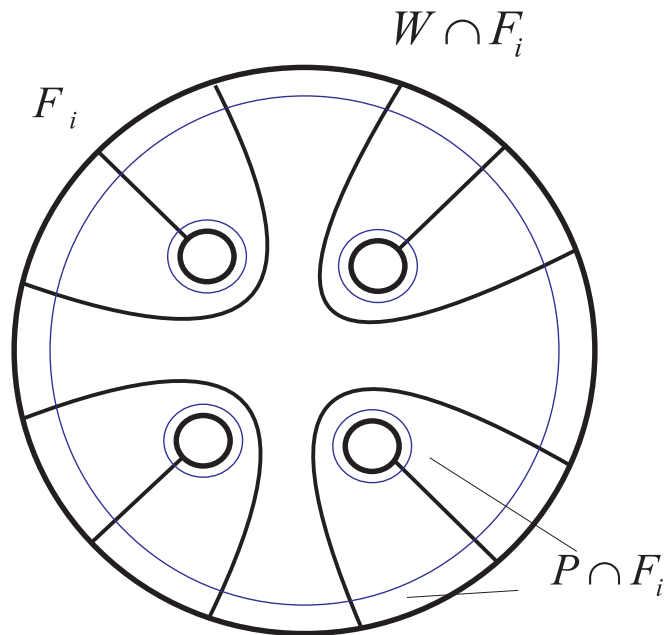


Fig. 9 $P \cap F_i$

330 **Proof.** By Lemma 5.1 and the argument of Lemma 4.2 and Lemma 4.3 in [4], the lemma holds. \square

Lemma 5.4. Each component of $P \cap N_0$ is isotopic to $F - (\bigcup E_i)$ or A_{a_i} . Furthermore the pattern of $P \cap F_n$ is as shown in Fig. 9, and $v(s_n) = v(s_i)$ (resp. $v(r_n^m) = v(r_i^m)$ for $m = 1, 2, 3, 4$) for $i = 1, 2, \dots, n-1$.

335 **Proof.** By Lemma 5.2 and Lemma 5.3, $P \cap F_i$ is as in Fig. 9 for $i = 1, 2, \dots, n-1$, and $P \cap F_n$ is as in one of Fig. 8.

Claim 3. Suppose S is a component of $P \cap N_0$, and ∂S has a component in the isotopic class of r_i^m for $i = 1, 2, \dots, n$ and $m = 1, 2, 3, 4$, then S must be an annular which is parallel to some A_{a_i} .

340 **Proof.** Suppose S is such a component of $P \cap N_0$ and S has a boundary component which is innermost and parallel to $\partial N(a_i) \cap F_j$ on F_j . Then $S \cap D_{a_i}$ is a span arc which cuts a disk from D_{a_i} . This disk severs Lemma 1.2, and hence S is parallel to A_{a_i} . See Fig. 10.

\square

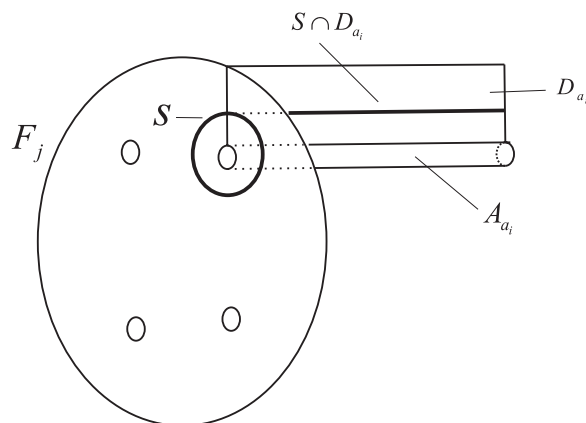


Fig. 10 $S \cap D_{a_i}$

By Claim 3, $v(r_n^m) = v(r_i^m)$ for $i = 1, 2, \dots, n-1$, $m = 1, 2, 3, 4$. By removing all the annular components of $P \cap N_0$, we can suppose $v(r_i^m) = 0$ for $i = 1, 2, \dots, n$, $m = 1, 2, 3, 4$.

Cutting N_0 along $\bigcup D_{a_i}$, the resulting manifold $N'_0 = N_0 - \text{int } N(\bigcup D_{a_i})$ is also homeomorphism to $F \times I$. Let $\partial_- N'_0 = \partial_- C$ and $\partial_+ N'_0 = \partial N'_0 - \partial_- N'_0$. Let $P' = P \cap N'_0$. By Lemma 1.4, P' is incompressible in N'_0 .

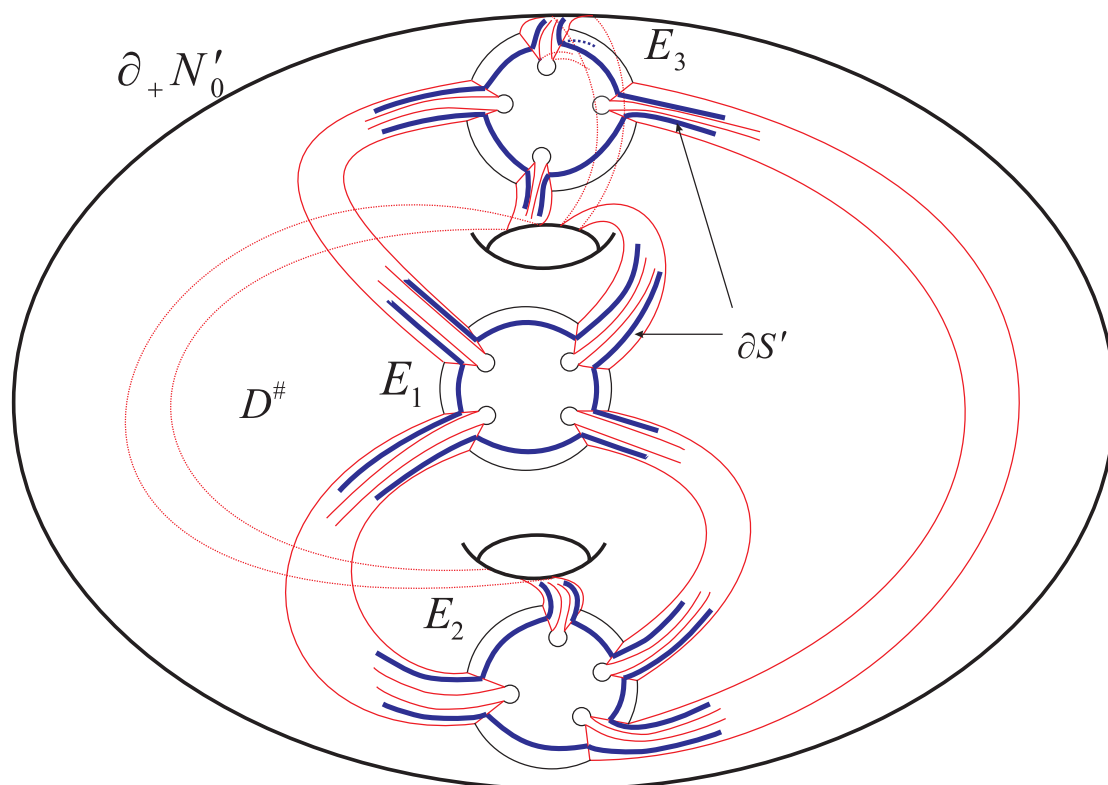


Fig. 11 $S \cap D_{a_i}$

Suppose that $v(s_n) \neq 0$. Then P' has a component S' with one boundary component as shown in Fig. 11. Recall the disk $D^\# = F - (\bigcup E_i) - (\bigcup a_i)$ in Proposition 2.1 (1). Since $\partial S'$ has a component which is isotopic to $\partial D^\#$, and that component bounds a disk in $\partial_+ N_0$

which is just outside of $D^\#$. By Lemma 1.1, S' is a disk.

360 Attaching back N_0 from N'_0 , S' becomes to S , which is a component of $P \cap N_0$ and it is isotopic to $F - (\bigcup E_i)$.

By removing all the components of $P \cap N_0$ which are isotopic to $F - (\bigcup E_i)$, we can suppose $v(s_n) = 0$. We also suppose that $v(x_n^1) > 0$ and $v(x_n^2) > 0$. Let c^m be the outermost component of $P \cap F_n$ in x_n^m for $m = 1, 2$.

365 Let p be the component of $\partial P'$ which is derive from c^1 and c^2 . See Fig.12.

There are two arcs a and b on F_n and two sub arcs c and d of c^1 and c^2 respectively, such that a, b, c and d bound a disk D^* on F_n . See Fig.12.

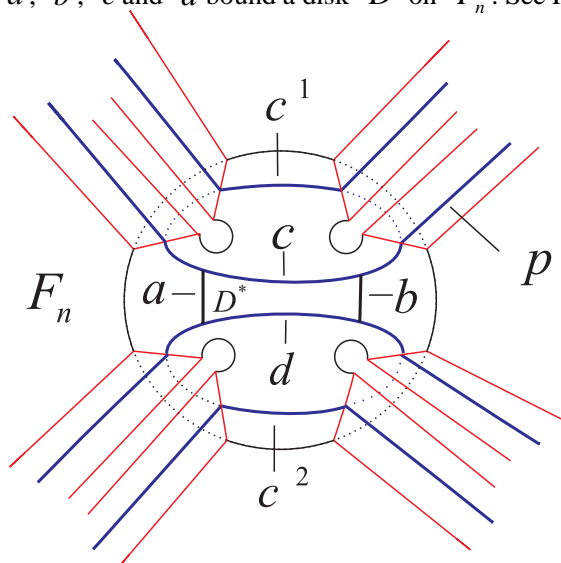


Fig. 12 $S \cap D_{a_i}$

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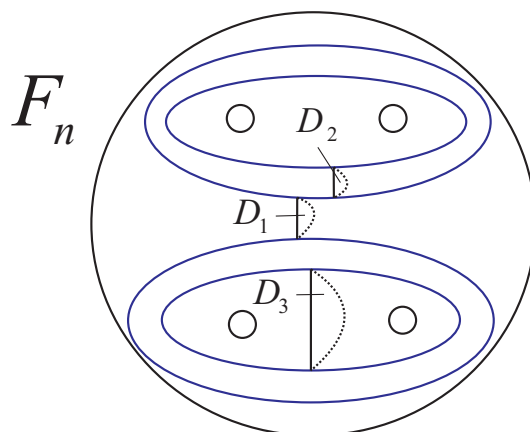
By the above discussion, we know that $(p - c - d) \cup a \cup b$ bounds a disk $\overline{D^\#}$ on $\partial_+ N_0$ which is just adjacent to $D^\#$. Let $A^\# = \overline{D^\#} \cup D^*$. Then $A^\#$ is an annular or a Mobius band with its boundary is p . Since $A^\#$ is a sub surface of an orientable surface $\partial_+ N'_0$, it must be an annular. Hence p contains two closed curves on $\partial_+ N'_0$, and they bounds an annular on $\partial_+ N'_0$.

375

By the same argument, we can prove that all the boundary components $P' \cap N'_0$ are parallel to p . By Lemma 1.3, each component of $P' \cap N'_0$ is parallel to $\partial_+ N'_0$. Then, we can always find ∂ -compressing disks D_1, D_2 and D_3 of P' as shown in Fig. 13.

By doing a 2-surgery of P along the possible ∂ -compressing disks as one of D_1, D_2 or D_3 , we can obtain a new incompressible surface P^* in C_k such that $C(P^*) < C(P)$. It is a contradiction.

380

Fig. 13 ∂ -compressing disks of S

Thus we have complete the proof of Lemma 6.4. \square

Lemma 5.5. Each component of $P \cap N_n$ is isotopic to one of, $A_{b_{2n}}$ or $A_{b_{2n-1}}$.

Proof. By Lemma 5.4, $P \cap F_n$ has the same pattern as $P \cap F_i$ for $i = 1, 2, \dots, n-1$. By the same proof of Lemma 4.5 in [4], this lemma holds. \square

Proposition 5.6. C_k contains no essential closed surfaces.

Proof. This follows from Lemma 5.3, Lemma 5.4 and Lemma 5.5. \square

The proof of Theorem 1. By Proposition 2.3, Proposition 3.4, Proposition 4.1 and Proposition 5.6, Theorem 1 holds.

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压缩体中的简单，极小纽结

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摘要: 在一个三维流形中, 如果一个纽结的补中不含有闭的本质不可压缩曲面, 就说这个纽结是小的纽结。在三维流形上的拓扑学的研究中, 一个问题就是: 判断在某个流形中是否含有小的纽结。Lopez 猜想说: 任何一个闭的小的三维流形中必定含有小的纽结。邱瑞锋和王诗宸证明了在亏格大于等于 2 的柄体中, 一定存在简单的、小的纽结。作为这一结果的推广: 作者在本文中证明了在某些压缩体中也存在简单的、小的纽结。设 F 是一个亏格至少为 2 的连通闭曲面, 设 C 是通过往 $F \times I$ 上添加 $2g(F)-1$ 个 1-把柄后得到的压缩体, 那么 C 中一定存在简单的、小的纽结。

关键词: 简单纽结; 极小纽结; 不可压缩曲面

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