随机激励下拟部分可积 Hamilton 系统的最优有界控制[1]

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摘 要：本文提出了一种基于拟部分可积 Hamilton 系统随机平均法和随机动态规划原理的最优有界控制策略。利用拟部分可积 Hamilton 系统随机平均法将受控的拟部分可积 Hamilton 系统化成一组部分平均的 Itô 随机微分方程；利用随机动态规划原理为系统响应最小化问题建立一个动态规划方程；在控制力为有界的条件下，可从动态规划方程与控制约束确定出最优控制力而无需求解该动态规划方程；受控系统的响应是通过求解与完全平均的 Itô 随机微分方程相应的 FPK 方程得到。最后用一个例子详细地阐述这个随机最优控制策略的具体实施过程与有效性。

关键词：非线性系统,随机激励,随机平均法,随机最优控制,动态规划原理

1. 引言

现实中很多受控系统都承受着随机载荷的作用，系统状态常从带有噪声的测量值估计得到，因此研究随机最优控制具有巨大的实际意义。随机最优控制的数学理论已经相当成熟了[1,2]。解决随机最优控制问题的一个主要方法是 Bellman 动态规划。迄今为止，随机最优控制理论主要应用于经济学，尤其是金融问题。在工程领域，直至最近只有线性二次高斯(LQG)控制策略得到了广泛的应用。近年来，本文第二作者和他的合作者们[3,4]提出了一种基于拟 Hamilton 系统随机平均法[5-7]和随机动态规划原理[1,2]的非线性随机最优控制策略，该策略已推广于部分可观测线性和非线性系统的最优控制[8,9]，非线性系统的半主动控制[10,11]，拟 Hamilton 系统的随机稳定化[12-14]及首次穿越损坏最小化[15-17]。研究结果表明，这种策略更有效，效率更高。更重要的是在控制策略中使用随机平均法简化了动态规划方程，减

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少了该方程的维数，并使之具有古典解。因此，这种策略具有很好的应用前景，值得进一步的研究。与此同时，胞映射法被用于求解非线性系统的随机最优控制中动态规划方程[18-20]。

文献[3,4]中所提出的控制策略中，控制力都是无约束的。然而，实践中，控制力的幅值往往是有界的，比如，作动器的饱和。最近，外激高斯白噪声作用下的线性系统的最优有界控制已有研究，并提出了动态规划方程的混合求解方法 [21-23]。本文基于拟部分可积 Hamilton 系统的随机平均法[7]和随机动态规划原理[1,2,24]提出了拟部分可积 Hamilton 系统的最优有界控制策略。最优控制力由动态规划方程与控制约束确定，通过解完全平均的 FPK 方程得到最优控制系统的响应。在算例中给出了一个具体系统稳态响应最小化问题，并给出了应用本文提出的随机最优有界控制策略解决这个问题的详细过程，最后用一些数值计算结果评价了这一控制策略的实施效果。

2. 拟部分可积 Hamilton 系统的随机平均法

考虑一个随机激励下 n 自由度受控拟 Hamilton 系统，系统运动方程为

$$\begin{aligned}
&\dot{Q}_i = \frac{\partial H'}{\partial P_i} \\
&\dot{P}_i = -\frac{\partial H'}{\partial Q_i} - \varepsilon c_{ij}(\boldsymbol{Q},\boldsymbol{P})\frac{\partial H'}{\partial P_j} + u_i(\boldsymbol{Q},\boldsymbol{P}) + \varepsilon^{1/2} f_{ik}(\boldsymbol{Q},\boldsymbol{P})\xi_k(t) \\
&i, j = 1, 2, \cdots, n; k = 1, 2, \cdots, m
\end{aligned} \tag{1}$$

式中 Q_i 和 P_i 分别表示广义位移和广义动量；$H' = H'(\boldsymbol{Q},\boldsymbol{P})$ 为 Hamilton 函数，连续且二阶可微；εc_{ij} 和 $\varepsilon^{1/2} f_{ik}$ 分别表示拟线性阻尼系数和随机激励的振幅；$u_i = u_i(\boldsymbol{Q},\boldsymbol{P})$ 是 ε 量级的反馈控制力；$\xi_k(t)$ 表示随机过程。

假设 $\xi_k(t)$ 是高斯白噪声，其相关函数为 $E[\xi_k(t)\xi_l(t+\tau)] = 2D_{kl}\delta(\tau)$。(1)式可以模型化为 Stratonovich 微分方程。通过加上 Wong-Zakai 修正项，把它们分成保守部分与耗散部分，并分别与 $-\partial H'/\partial Q_i$ 和 $-\varepsilon c_{ij}(\boldsymbol{Q},\boldsymbol{P})\partial H'/\partial P_j$ 合并，最后得到如下 Itô 随机微分方程：

$$\begin{aligned}
&dQ_i = \frac{\partial H}{\partial P_i} dt \\
&dP_i = -\{\frac{\partial H}{\partial Q_i} - \varepsilon m_{ij}(\boldsymbol{Q},\boldsymbol{P})\frac{\partial H}{\partial P_j} - u_i(\boldsymbol{Q},\boldsymbol{P})\}dt + \varepsilon^{1/2}\sigma_{ik}(\boldsymbol{Q},\boldsymbol{P})dB_k(t) \\
&i, j = 1, 2, \cdots, n; k = 1, 2, \cdots, m
\end{aligned} \tag{2}$$

式中 $H = H(\boldsymbol{Q}, \boldsymbol{P})$ 是经过修正的 Hamilton 函数， $m_{ij} = m_{ij}(\boldsymbol{Q}, \boldsymbol{P})$ 是经过修正的阻尼系数；$\boldsymbol{\sigma\sigma}^T = 2\boldsymbol{fDf}^T$， $\boldsymbol{D} = [D_{kl}]$， $B_k(t)$ 是标准 Wiener 过程。

假设与(2)相应的 Hamilton 系统是部分可积且非内共振[7]，有 $r(1<r<n)$ 个独立对合的首次积分 $H_1, H_2, \cdots, H_r$。为确定起见，设 Hamilton 函数形为

$$H(\boldsymbol{q}, \boldsymbol{p}) = \sum_{\eta=1}^{r-1} H_\eta(\boldsymbol{p_1}, \boldsymbol{q_1}) + H_r(\boldsymbol{p_2}, \boldsymbol{q_2}) \tag{3}$$

式中 $\boldsymbol{q}_1 = [q_1, q_2, \cdots q_{r-1}]^T$， $\boldsymbol{p}_1 = [p_1, p_2, \cdots p_{r-1}]^T$， $\boldsymbol{q}_2 = [q_r, q_{r+1}, \cdots q_n]^T$，$\boldsymbol{p}_2 = [p_r, p_{r+1}, \cdots p_n]^T$。对可积部分引入作用－角变量

$$I_\eta = I_\eta(\boldsymbol{q}_1, \boldsymbol{p}_1),\ \ \Theta_\eta = \Theta_\eta(\boldsymbol{q}_1, \boldsymbol{p}_1)\ ,\ \eta = 1, 2, \cdots r-1 \tag{4}$$

运用 Itô 微分公式可以从(2)式得到 I_η， Θ_η 和 H_r 所满足的 Itô 随机微分方程：

$$\begin{aligned}
dI_\eta &= \varepsilon\left(-m_{\eta' j}\frac{\partial H}{\partial P_j}\frac{\partial I_\eta}{\partial P_{\eta'}} + \frac{1}{2}\sigma_{\eta' k}\sigma_{\eta'' k}\frac{\partial^2 I_\eta}{\partial P_{\eta'}\partial P_{\eta''}} + \left\langle u_{\eta'}\frac{\partial I_\eta}{\partial P_{\eta'}}\right\rangle\right)dt + \varepsilon^{1/2}\frac{\partial I_\eta}{\partial P_{\eta'}}\sigma_{\eta' k}dB_k(t)\\
d\Theta_\eta &= \left(\omega_n - \varepsilon m_{\eta' j}\frac{\partial H}{\partial P_j}\frac{\partial \Theta_\eta}{\partial P_{\eta'}} + \frac{\varepsilon}{2}\sigma_{\eta' k}\sigma_{\eta'' k}\frac{\partial^2 \Theta_\eta}{\partial P_{\eta'}\partial P_{\eta''}} + \left\langle u_{\eta'}\frac{\partial \Theta_\eta}{\partial P_{\eta'}}\right\rangle\right)dt\\
&\quad + \varepsilon^{1/2}\frac{\partial \Theta_\eta}{\partial P_{\eta'}}\sigma_{\eta' k}dB_k(t)
\end{aligned} \tag{5}$$

$$dH_r = \varepsilon\left(-m_{\rho j}\frac{\partial H}{\partial P_j}\frac{\partial H_r}{\partial P_\rho} + \frac{1}{2}\sigma_{\rho k}\sigma_{\rho' k}\frac{\partial^2 H_r}{\partial P_\rho \partial P_{\rho'}} + \left\langle u_\rho \frac{\partial H_r}{\partial P_\rho}\right\rangle\right)dt + \varepsilon^{1/2}\frac{\partial H_r}{\partial P_\rho}\sigma_{\rho k}dB_k(t)$$

$$j = 1, 2, \cdots, n; \eta, \eta', \eta'' = 1, 2, \cdots, r-1; \rho, \rho' = r, r+1, \cdots, n; k = 1, 2, \cdots, m$$

运用拟部分可积 Hamilton 系统随机平均法[7]，最后得到平均后的 Itô 随机微分方程为

$$\begin{aligned}
dI_\eta &= [\bar{m}_\eta(\boldsymbol{I}, H_r) + \left\langle u_{\eta'}\frac{\partial I_\eta}{\partial P_{\eta'}}\right\rangle]dt + \bar{\sigma}_{\eta k}(\boldsymbol{I}, H_r)dB_k(t)\\
dH_r &= [\bar{m}_r(\boldsymbol{I}, H_r) + \left\langle u_\rho \frac{\partial H_r}{\partial P_\rho}\right\rangle]dt + \bar{\sigma}_{rk}(\boldsymbol{I}, H_r)dB_k(t)\\
&\eta', \eta = 1, 2, \cdots, r-1;\quad k = 1, 2, \cdots, m;\quad \rho = r, r+1 \cdots n;
\end{aligned} \tag{6}$$

其中平均漂移和扩散系数为

$$
\begin{aligned}
&\overline{m}_{\eta}(\boldsymbol{I},H)=\varepsilon\left\langle -m_{\eta' j}\frac{\partial H}{\partial P_j}\frac{\partial I_\eta}{\partial P_{\eta'}}+\frac{1}{2}\sigma_{\eta' k}\sigma_{\eta'' k}\frac{\partial^2 I_\eta}{\partial P_{\eta'}\partial P_{\eta''}}\right\rangle \\
&\overline{m}_{r}(\boldsymbol{I},H)=\varepsilon\left\langle -m_{\rho j}\frac{\partial H}{\partial P_j}\frac{\partial H_r}{\partial P_{\rho}}+\frac{1}{2}\sigma_{\rho k}\sigma_{\rho' k}\frac{\partial^2 H_r}{\partial P_{\rho}\partial P_{\rho'}}\right\rangle \\
&\overline{\sigma}_{\eta k}\overline{\sigma}_{\overline{\eta} k}(\boldsymbol{I},H)=\varepsilon\left\langle \sigma_{\eta' k}\sigma_{\eta'' k}\frac{\partial I_\eta}{\partial P_{\eta'}}\frac{\partial I_{\overline{\eta}}}{\partial P_{\eta''}}\right\rangle \\
&\overline{\sigma}_{\eta k}\overline{\sigma}_{r k}(\boldsymbol{I},H)=\varepsilon\left\langle \sigma_{\eta' k}\sigma_{\rho k}\frac{\partial I_\eta}{\partial P_{\eta'}}\frac{\partial H_r}{\partial P_{\rho}}\right\rangle \\
&\overline{\sigma}_{r k}\overline{\sigma}_{r k}(\boldsymbol{I},H)=\varepsilon\left\langle \sigma_{\rho k}\sigma_{\rho' k}\frac{\partial H_r}{\partial p_{\rho}}\frac{\partial H_r}{\partial p_{\rho'}}\right\rangle \\
&\rho'=r,r+1,\cdots n,\quad \overline{\eta},\eta''=1,2,\cdots r-1
\end{aligned}
\tag{7}
$$

式中

$$
\begin{aligned}
&\langle\bullet\rangle=\frac{1}{(2\pi)^{r-1}T(H_r)}\int_{\Omega_1}\int_0^{2\pi}[\bullet/\frac{\partial H_r}{\partial p_r}]d\boldsymbol{\theta}dq_r\cdots dq_n dp_{r+1}\cdots dp_n \\
&T(H_r)=\int_{\Omega_1}[1/\frac{\partial H_r}{\partial p_r}]dq_r\cdots dq_n dp_{r+1}\cdots dp_n \\
&\Omega_1=\{(q_r,\cdots q_n,p_{r+1},\cdots p_n)\,|\,H_r(q_r,\cdots q_n,0,p_{r+1},\cdots p_n)\le H_r\} \\
&\boldsymbol{I}=[I_1,I_2,\cdots I_{r-1}]^T\quad \boldsymbol{\theta}=[\theta_1,\theta_2,\cdots\theta_{r-1}]^T
\end{aligned}
\tag{8}
$$

特别地，当 Hamilton 函数具有如下形式

$$
\begin{aligned}
&H_\eta=p_\eta^2/2+U_\eta(q_\eta),\quad \eta=1,2,\cdots,r-1 \\
&H_r=\sum_{\rho=r}^{n}p_\rho^2/2+U_r(\boldsymbol{q}_2,\boldsymbol{p}_2)
\end{aligned}
\tag{9}
$$

时，引入作用一角变量

$$
I_\eta=f_\eta(H_\eta),\quad \theta_\eta=\omega_\eta(I_\eta)+\delta_\eta \tag{10}
$$

式中

$$
\omega_\eta(I_\eta)=\frac{dH_\eta}{dI_\eta}=\frac{df_\eta^{-1}(I_\eta)}{dI_\eta} \tag{11}
$$

由于 H_η 和 I_η 一一对应，与 θ_η 无关，可用 H_η 替代(6)式中的 I_η 得到如下关于 H_η，H_r 的 Itô 微分方程：

$$dH_\eta=\left[\bar m_\eta(\boldsymbol H)+\left\langle u_{\eta'}P_\eta\right\rangle\right]dt+\bar\sigma_{\eta k}(\boldsymbol H)dB_k(t)$$
$$dH_r=\left[\bar m_r(\boldsymbol H)+\left\langle u_\rho P_\rho\right\rangle\right]dt+\bar\sigma_{rk}(\boldsymbol H)dB_k(t) \tag{12}$$
$$\eta=1,2,\cdots,r-1;\quad k=1,2,\cdots,m;$$

其中 $\boldsymbol H=[H_1,H_2\cdots H_r]$，漂移和扩散系数为

$$\bar m_\eta(\boldsymbol H)=\left\langle -m_{\eta j}P_jP_\eta+\sigma_{\eta k}\sigma_{\eta k}/2\right\rangle$$
$$\bar m_r(\boldsymbol H)=\left\langle -m_{\rho j}P_jP_\rho+\sigma_{\rho k}\sigma_{\rho k}/2\right\rangle$$
$$\bar\sigma_{\eta k}\bar\sigma_{\bar\eta k}(\boldsymbol H)=\left\langle P_\eta P_{\bar\eta}\sigma_{\eta k}\sigma_{\bar\eta k}\right\rangle \tag{13}$$
$$\bar\sigma_{rk}\bar\sigma_{rk}(\boldsymbol H)=\left\langle P_\rho^2\sigma_{\rho k}\sigma_{\rho k}\right\rangle$$
$$\bar\sigma_{\eta k}\bar\sigma_{rk}(\boldsymbol H)=\left\langle P_\eta P_\rho\sigma_{\eta k}\sigma_{\rho k}\right\rangle$$

式中

$$\langle\bullet\rangle=\frac{1}{T(H_1)\cdots T(H_r)}\int_{\Omega_1}\oint\left[\cdot\Big/\frac{\partial H_r}{\partial p_r}\right]\frac{dq_1\cdots dq_n dp_{r+1}\cdots dp_n}{p_1\cdots p_{r-1}} \tag{14}$$

3. 最优控制规律

假设施加于系统的控制力受到如下的有界约束

$$|u_i|\le b_i,\quad b_i>0,\quad i=1,2,\cdots,n \tag{15}$$

所考虑的性能指标为

$$J=\lim_{T\to\infty}\frac{1}{T}[\int_0^T f(\boldsymbol I(s),H_r(s))ds] \tag{16}$$

运用随机动态规划原理[24]就可以建立如下动态规划方程：

$$\min_{|u_i|\le b_i}\{[\bar m_\eta+<u_{\eta'}\frac{\partial I_\eta}{\partial P_{\eta'}}>]\frac{\partial V}{\partial I_\eta}+[\bar m_r+<u_\rho\frac{\partial H_r}{\partial P_\rho}>]\frac{\partial V}{\partial H_r}$$
$$+\sigma_{\eta l}\sigma_{rl}\frac{\partial^2V}{\partial I_\eta\partial H_r}+\frac12\sigma_{\eta l}\sigma_{\bar\eta l}\frac{\partial^2V}{\partial I_\eta\partial I_{\bar\eta}}+\frac12\sigma_{rl}\sigma_{rl}\frac{\partial^2V}{\partial H_r^2}+f(\boldsymbol I,H_r)\}=\gamma \tag{17}$$

式中 $V=V(\boldsymbol I,H_r)$ 是值函数，

$$\gamma=\lim_{T\to\infty}\frac1T[\int_0^T f(\boldsymbol I(s),H_r(s))ds] \tag{18}$$

表示最优平均成本。由(17)式和(15)式可以解得最优控制规律：

$$u_\eta^* = -b_\eta\, sgn(\frac{\partial I_\eta}{\partial P_\eta}\frac{\partial V}{\partial I_\eta})$$
$$u_\rho^* = -b_\rho\, sgn(\frac{\partial H_r}{\partial P_\rho}\frac{\partial V}{\partial H_r}) \qquad (19)$$
$$\eta = 1,2,\cdots r-1;\quad \rho = r, r+1,\cdots n$$

这里得$'sgn'$表示符号函数。取 $f(\boldsymbol{I}, H_r)$ 使得 $\partial V/\partial I_\eta$ 和 $\partial V/\partial H_r$ 都为正，那么，最优控制规律的表达式(19)简化为

$$u_\eta^* = -b_\eta\, sgn(\frac{\partial I_\eta}{\partial P_\eta})$$
$$u_\rho^* = -b_\rho\, sgn(\frac{\partial H_r}{\partial P_\rho}) \qquad (20)$$
$$\eta = 1,2,\cdots r-1;\quad \rho = r, r+1,\cdots n$$

从(20)式可以看出最优控制为干摩擦或者称为 bang-bang 控制，u_η^* 和 u_ρ^* 具有常幅值，且分别在 $\partial I_\eta/\partial P_\eta = 0$ 和 $\partial H_r/\partial P_r = 0$ 时改变方向。

对于(12)式所示的平均 Itô 随机微分方程，类似可求得最优控制力为：

$$u_\eta^* = -b_\eta\, sgn(\frac{\partial H_\eta}{\partial P_\eta})$$
$$u_\rho^* = -b_\rho\, sgn(\frac{\partial H_r}{\partial P_\rho}) \qquad (21)$$
$$\eta = 1,2,\cdots r-1;\quad \rho = r, r+1,\cdots n$$

对有限区间 $t \in [0, t_f]$ 上的有界控制，设所考虑的性能指标为

$$J_1 = E[\int_0^{t_f} f_1(\boldsymbol{I}(s), H_r(s))ds + g_1(\boldsymbol{I}(t_f), H_r(t_f))] \qquad (22)$$

引入值函数

$$V = \min_{|u_i|<b_i} E[\int_t^{t_f} f_1(\boldsymbol{I}(s), H_r(s))ds + g_1(\boldsymbol{I}(t_f), H_r(t_f))] \qquad (23)$$

可导出动态规划方程

$$\frac{\partial V_1}{\partial t_1} = -\min_{|u_i|\le b_i}\{[\bar{m}_\eta + < u_{\eta'}\frac{\partial I_\eta}{\partial P_{\eta'}} >]\frac{\partial V_1}{\partial I_\eta} + [\bar{m}_r + < u_\rho \frac{\partial H_r}{\partial P_\rho} >]\frac{\partial V_1}{\partial H_r}$$

$$+\sigma_{\eta l}\sigma_{rl}\frac{\partial^2 V_1}{\partial I_\eta \partial H_r}+\frac{1}{2}\sigma_{\eta l}\sigma_{\bar{\eta} l}\frac{\partial^2 V_1}{\partial I_\eta \partial I_{\bar{\eta}}}+\frac{1}{2}\sigma_{rl}\sigma_{rl}\frac{\partial^2 V_1}{\partial H_r^2}+f(\boldsymbol{I},H_r)\} \tag{24}$$

可以由(24)式导得类似于(20)式的最优控制力。

4. 最优控制系统的响应

将(20)式中u_η^*和u_ρ^*代入(6)式取代u_η和u_ρ，并完成平均运算，就得到最优控制系统的完全平均的 Itô 随机微分方程

$$\begin{aligned}
dI_\eta &= \bar{\bar{m}}_\eta(\boldsymbol{I},H_r)dt+\bar{\sigma}_{\eta k}(\boldsymbol{I},H_r)dB_k(t)\\
dH_r &= \bar{\bar{m}}_r(\boldsymbol{I},H_r)dt+\bar{\sigma}_{rk}(\boldsymbol{I},H_r)dB_k(t)\\
\eta &= 1,2,\cdots,r-1;\quad k=1,2,\cdots,m;\quad \boldsymbol{I}=[I_1,I_2\cdots I_{r-1}]^T
\end{aligned} \tag{25}$$

其中

$$\begin{aligned}
\bar{\bar{m}}_\eta(\boldsymbol{I},H_r) &= \bar{m}_\eta(\boldsymbol{I},H_r)+\left\langle u_{\eta'}^*\frac{\partial I_\eta}{\partial P_{\eta'}}\right\rangle\\
\bar{\bar{m}}_r(\boldsymbol{I},H_r) &= \bar{m}_r(\boldsymbol{I},H_r)+\left\langle u_\rho^*\frac{\partial H_r}{\partial P_\rho}\right\rangle
\end{aligned} \tag{26}$$

与(25)式相对应的平均 FPK 方程为

$$\begin{aligned}
\frac{\partial p}{\partial t} &= -\frac{\partial}{\partial I_\eta}(a_\eta p)-\frac{\partial}{\partial H_r}(a_r p)+\frac{1}{2}\frac{\partial^2}{\partial I_\eta \partial I_{\bar{\eta}}}(b_{\eta\bar{\eta}}p)\\
&\quad +\frac{\partial^2}{\partial I_\eta \partial H_r}(b_{\eta r}p)+\frac{1}{2}\frac{\partial^2}{\partial H_r^2}(b_{rr}p)
\end{aligned} \tag{27}$$

式中

$$\begin{aligned}
a_\eta &= a_\eta(\boldsymbol{I},H_r)=\bar{\bar{m}}_\eta(\boldsymbol{I},H_r)\\
a_r &= a_r(\boldsymbol{I},H_r)=\bar{\bar{m}}_r(\boldsymbol{I},H_r)\\
b_{\eta\bar{\eta}} &= b_{\eta\bar{\eta}}(\boldsymbol{I},H_r)=\bar{\sigma}_{\eta k}\bar{\sigma}_{\bar{\eta} k}(\boldsymbol{I},H_r)\\
b_{\eta r} &= b_{\eta r}(\boldsymbol{I},H_r)=\bar{\sigma}_{\eta k}\bar{\sigma}_{rk}(\boldsymbol{I},H_r)\\
b_{rr} &= b_{rr}(\boldsymbol{I},H_r)=\bar{\sigma}_{rk}\bar{\sigma}_{rk}(\boldsymbol{I},H_r)
\end{aligned} \tag{28}$$

其中$p=p(\boldsymbol{I},H_r,0\,|\,\boldsymbol{I_0},H_{r0})$表示转移概率密度，它的初始条件为

$$p(\boldsymbol{I},H_r,0\,|\,\boldsymbol{I_0},H_{r0})=\delta(\boldsymbol{I}-\boldsymbol{I_0})\delta(H_r-H_{r0}) \tag{29}$$

边界条件为

$$p, \partial p / \partial I_\eta, \partial p / \partial H_r \to 0 \quad \text{as} \quad |\boldsymbol{I}| \to \infty, H_r \to \infty \tag{30}$$

对于遍历控制问题，只求系统的稳态概率密度。此时，时间导数项$\partial p / \partial t = 0$。考虑到概率密度的非负性和边界条件(30)，假设 FPK 方程(27)的精确平稳解形为

$$p(\boldsymbol{I}, H_r) = C \exp[-\lambda(\boldsymbol{I}, H_r)] \tag{31}$$

式中$\lambda(\boldsymbol{I}, H_r)$是概率势，$C$ 归一化常数。将(31)代入没有时间导数项的 FPK 方程(27)中，就可得到$\lambda(\boldsymbol{I}, H_r)$和$p(\boldsymbol{I}, H_r)$。系统的广义位移和广义动量的联合稳态概率密度函数可以由下式得到[7]

$$p(\boldsymbol{q}, \boldsymbol{p}) = \left. \frac{p(\boldsymbol{I}, H_r)}{(2\pi)^{r-1} T(\boldsymbol{I}, H_r)} \right|_{H_r = H_r(\boldsymbol{q}, \boldsymbol{p}), \boldsymbol{I} = \boldsymbol{I}(\boldsymbol{p}.\boldsymbol{q})} \tag{32}$$

将(21)式中u_η^*和u_ρ^*代入(12)式取代u_η和u_ρ，并且完成平均运算，就得到系统完全平均后的 FPK 方程：

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial H_\eta}(a_\eta p) - \frac{\partial}{\partial H_r}(a_r p) + \frac{1}{2}\frac{\partial^2}{\partial H_\eta \partial H_{\bar{\eta}}}(b_{\eta\bar{\eta}} p)$$
$$+ \frac{\partial^2}{\partial H_\eta \partial H_r}(b_{\eta r} p) + \frac{1}{2}\frac{\partial^2}{\partial H_r^2}(b_{rr} p) \tag{33}$$

其中

$$\begin{aligned}
a_\eta &= a_\eta(\boldsymbol{H}) = \bar{\bar{m}}_\eta(\boldsymbol{H}) = \bar{m}_\eta(\boldsymbol{H}) + \left\langle u_\eta^* P_\eta \right\rangle \\
a_r &= a_r(\boldsymbol{H}) = \bar{\bar{m}}_r(\boldsymbol{H}) = \bar{m}_r(\boldsymbol{H}) + \left\langle u_\rho^* P_\rho \right\rangle \\
b_{\eta\bar{\eta}} &= b_{\eta\bar{\eta}}(\boldsymbol{H}) = \bar{\sigma}_{\eta k} \bar{\sigma}_{\bar{\eta} k}(\boldsymbol{H}) \\
b_{\eta r} &= b_{\eta r}(\boldsymbol{H}) = \bar{\sigma}_{\eta k} \bar{\sigma}_{rk}(\boldsymbol{H}) \\
b_{rr} &= b_{rr}(\boldsymbol{H}) = \bar{\sigma}_{rk} \bar{\sigma}_{rk}(\boldsymbol{H})
\end{aligned} \tag{34}$$

FPK 方程(33)的精确稳态解假设成如下形式[7]

$$p(\boldsymbol{H}) = C \exp[-\lambda(\boldsymbol{H})] \tag{35}$$

系统的广义位移和广义动量的联合概率密度为

$$p(\boldsymbol{q}, \boldsymbol{p}) = \left. \frac{p(\boldsymbol{H})}{T(H_1) \cdots T(H_r)} \right|_{H_\eta = H_\eta(q_\eta, p_\eta), H_r = H_r(\boldsymbol{q}, \boldsymbol{p})} \tag{36}$$

在得到受最优控制系统(1)的响应的稳态概率密度和联合概率密度之后，关于稳态响应的其他统计量(比如 $p(q_i)$， $E[Q_i]$等)就可以通过(32)式和(36)式计算得到。而未控系统的概率密度和其他统计量则可以通过令 $u_i^*=0$ 得到。

5. 算例

作为上述理论的一个应用的例子，考虑一个 4 自由度受控拟 Hamilton 系统，其运动方程如下

$$\begin{aligned}
&\dot{Q}_1 = P_1\\
&\dot{P}_1 = -\omega_1^2 Q_1 - \left[\alpha_{10}+\alpha_{11}P_1^2+\alpha_{12}P_2^2+\alpha_{13}P_3^2+\alpha_{14}P_4^2\right.\\
&\qquad \left. +(\alpha_{13}+\alpha_{14})U(Q_3,Q_4)\right]P_1+u_1+\xi_1(t)\\
&\dot{Q}_2 = P_2\\
&\dot{P}_2 = -\omega_2^2 Q_2 - \left[\alpha_{20}+\alpha_{21}P_1^2+\alpha_{22}P_2^2+\alpha_{23}P_3^2+\alpha_{24}P_4^2\right.\\
&\qquad \left. +(\alpha_{23}+\alpha_{24})U(Q_3,Q_4)\right]P_2+u_2+\xi_2(t)\\
&\dot{Q}_3 = P_3\\
&\dot{P}_3 = -\partial U/\partial Q_3 - \left[\alpha_{30}+\alpha_{31}P_1^2+\alpha_{32}P_2^2+\alpha_{33}P_3^2+\alpha_{34}P_4^2\right.\\
&\qquad \left. +(1/2)(\alpha_{34}+3\alpha_{33})U(Q_3,Q_4)\right]P_3+u_3+\xi_3(t)\\
&\dot{Q}_4 = P_4\\
&\dot{P}_4 = -\partial U/\partial Q_4 - \left[\alpha_{40}+\alpha_{41}P_1^2+\alpha_{42}P_2^2+\alpha_{43}P_3^2+\alpha_{44}P_4^2\right.\\
&\qquad \left. +(1/2)(\alpha_{43}+3\alpha_{44})U(Q_3,Q_4)\right]P_4+u_4+\xi_4(t)
\end{aligned} \tag{37}$$

其中

$$U(Q_3,Q_4)=k\left(\omega_3^2Q_3^2+\omega_4^2Q_4^2\right)^3/6 \tag{38}$$

ω_1， ω_2， k 为正常数；$\xi_k(t)$ 是强度为 $2D_k$ 的独立 Gauss 白噪声；u_i 为反馈控制力；设 α_{ij}，D_k， u_i 是同阶小量参数。该系统 Hamilton 函数 H 具有(9)式的形式

$$H=\sum_{\eta=1}^{2}H_\eta+H_3=\sum_{\eta=1}^{2}\omega_\eta I_\eta+H_3 \tag{39}$$

其中

$$H_\eta=\left(p_\eta^2+\omega_\eta^2q_\eta^2\right)/2,\quad H_3=\left(p_3^2+p_4^2\right)/2+U(q_3,q_4) \tag{40}$$

$U(q_3,q_4)$ 为不可分离，所以(37)是拟部分可积 Hamilton 系统，它具有三个独立的首次积分 H_1，H_2，H_3。假设系统(37)为非内共振。由于 Hamilton 函数 H 具有(9)式的形式，所以部分平均 Itô 随机微分方程具有式(12)所示形式，即

$$\begin{aligned}
dH_1 &= [\bar{m}_1(\boldsymbol{H}) + \langle u_1 P_1 \rangle]dt + \bar{\sigma}_{11}(\boldsymbol{H})dB_1 \\
dH_2 &= [\bar{m}_2(\boldsymbol{H}) + \langle u_2 P_2 \rangle]dt + \bar{\sigma}_{22}(\boldsymbol{H})dB_2 \\
dH_3 &= [\bar{m}_3(\boldsymbol{H}) + \langle u_3 P_3 + u_4 P_4 \rangle]dt + \bar{\sigma}_{33}(\boldsymbol{H})dB_3
\end{aligned} \tag{41}$$

漂移和扩散系数形如(13)式，即

$$\begin{aligned}
&T(H_\eta) = 2\pi/\omega_\eta,\ T(H_3) = \frac{2\pi^2}{\omega_3\omega_4}\sqrt[3]{\frac{6H_3}{k}} \\
&\bar{m}_1(\boldsymbol{H}) = -\Big[\alpha_{10}H_1 + 3\alpha_{11}H_1^2/2 + \alpha_{12}H_1H_2 \\
&\qquad\qquad + (\alpha_{13}+\alpha_{14})H_1H_3\Big] + D_1 \\
&\bar{m}_2(\boldsymbol{H}) = -\Big[\alpha_{20}H_2 + 3\alpha_{22}H_2^2/2 + \alpha_{21}H_1H_2 \\
&\qquad\qquad + (\alpha_{23}+\alpha_{24})H_2H_3\Big] + D_2 \\
&\bar{m}_3(\boldsymbol{H}) = -\Big[(\alpha_{30}+\alpha_{40}) + (\alpha_{31}+\alpha_{41})H_1 \\
&\quad + (\alpha_{32}+\alpha_{42})H_2 + (3\alpha_{33}+3\alpha_{44}+\alpha_{34}+\alpha_{43})H_3/2\Big]\frac{3H_3}{4} \\
&\qquad\qquad + D_3 + D_4 \\
&\bar{\sigma}_{11}^2(\boldsymbol{H}) = 2D_1H_1,\ \bar{\sigma}_{22}^2(\boldsymbol{H}) = 2D_2H_2 \\
&\bar{\sigma}_{33}^2(\boldsymbol{H}) = \frac{3}{2}(D_3+D_4)H_3,
\end{aligned} \tag{42}$$

如果控制力 u_i 受到形如(15)式那样的有界约束，那么最优控制力 u_i^* 就具有(21)式那样的形式。把最优控制力 u_i^* 代入方程(41)，并完成平均运算，最后得到如下完全平均之后的 Itô 随机微分方程

$$\begin{aligned}
dH_1 &= \bar{\bar{m}}_1(\boldsymbol{H})dt + \bar{\sigma}_{11}(\boldsymbol{H})dB_1 \\
dH_2 &= \bar{\bar{m}}_2(\boldsymbol{H})dt + \bar{\sigma}_{22}(\boldsymbol{H})dB_2 \\
dH_3 &= \bar{\bar{m}}_3(\boldsymbol{H})dt + \bar{\sigma}_{33}(\boldsymbol{H})dB_3
\end{aligned} \tag{43}$$

式中

$$\bar{\bar{m}}_1(\boldsymbol{H}) = [\bar{m}_1(\boldsymbol{H}) - \frac{2\sqrt{2}b_1}{\pi}\sqrt{H_1}]$$
$$\bar{\bar{m}}_2(\boldsymbol{H}) = [\bar{m}_2(\boldsymbol{H}) - \frac{2\sqrt{2}b_2}{\pi}\sqrt{H_2}] \tag{44}$$
$$\bar{\bar{m}}_3(\boldsymbol{H}) = [\bar{m}_3(\boldsymbol{H}) - \frac{2\sqrt{2}(b_3+b_4)}{3\pi}Beta(\frac{1}{3},\frac{3}{2})\sqrt{H_3}]$$

与(43)式相对应的 FPK 方程为

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial H_\eta}(a_\eta p) - \frac{\partial}{\partial H_r}(a_r p) + \frac{1}{2}\frac{\partial^2}{\partial H_\eta^2}(b_{\eta\eta} p) + \frac{1}{2}\frac{\partial^2}{\partial H_r^2}(b_{rr} p) \tag{45}$$
$$\eta = 1,2;\quad r = 3$$

其中

$$a_\eta = a_\eta(\boldsymbol{H}) = \bar{\bar{m}}_\eta(\boldsymbol{H}) \;,\quad a_r = a_r(\boldsymbol{H}) = \bar{\bar{m}}_r(\boldsymbol{H})$$
$$b_{\eta\eta} = b_{\eta\eta}(\boldsymbol{H}) = \bar{\sigma}^2_{\eta\eta}(\boldsymbol{H}) \;,\quad b_{rr} = b_{rr}(\boldsymbol{H}) = \bar{\sigma}^2_{rr}(\boldsymbol{H}) \tag{46}$$

Hamilton 函数 $\boldsymbol{H}$ 的初始条件为

$$p(\boldsymbol{H},0\,|\,\boldsymbol{H}_0) = \delta(\boldsymbol{H}-\boldsymbol{H}_0) \tag{47}$$

边界条件为

$$p, \partial p/\partial H_\eta, \partial p/\partial H_r \to 0, |\boldsymbol{H}| \to \infty \tag{48}$$

假设方程(45)平稳解的形式为

$$p(\boldsymbol{H}) = C\exp[-\lambda(\boldsymbol{H})] \tag{49}$$

将(49)式代入(45)式，其中 $\partial p/\partial t = 0$，可以得到

$$\frac{\partial\lambda}{\partial H_1}D_1H_1 = D_1 - a_1$$
$$\frac{\partial\lambda}{\partial H_2}D_2H_2 = D_2 - a_2 \tag{50}$$
$$\frac{3}{4}\frac{\partial\lambda}{\partial H_3}(D_3+D_4)H_3 = \frac{3}{4}(D_3+D_4) - a_3$$

假设 $\lambda(\boldsymbol{H})$ 满足相容性条件 $\partial^2\lambda/\partial H_i\partial H_j = \partial^2\lambda/\partial H_i\partial H_j$， $i, j = 1,2,3$，即

$$\alpha_{12}/D_1 = \alpha_{21}/D_2 \;,\quad (\alpha_{13}+\alpha_{14})/D_1 = (\alpha_{31}+\alpha_{41})/(D_3+D_4)$$
$$(\alpha_{23}+\alpha_{24})/D_2 = (\alpha_{32}+\alpha_{42})/(D_3+D_4) \tag{51}$$

则由(50)式解得 $\lambda(\boldsymbol{H})$

$$\lambda(\boldsymbol{H})=\left[\alpha_{10}H_1+3\alpha_{11}H_1^2/4+\alpha_{12}H_1H_2+(\alpha_{13}+\alpha_{14})H_1H_3\right]/D_1$$
$$+\left[\alpha_{20}H_2+3\alpha_{22}H_2^2/4+\alpha_{21}H_1H_2+(\alpha_{23}+\alpha_{24})H_2H_3\right]/D_2$$
$$-\ln H_3^{1/3}+[(\alpha_{30}+\alpha_{40})H_3+(\alpha_{31}+\alpha_{41})H_1H_3+(\alpha_{32}+\alpha_{42})H_2H_3$$
$$+(3\alpha_{33}+3\alpha_{44}+\alpha_{34}+\alpha_{43})H_3^2/4]/(D_3+D_4)$$
$$+\frac{4\sqrt{2}b_1}{\pi D_1}\sqrt{H_1}+\frac{4\sqrt{2}b_2}{\pi D_2}\sqrt{H_2}+\frac{16\sqrt{2}(b_3+b_4)}{9\pi(D_3+D_4)}Beta(\frac{1}{3},\frac{3}{2})\sqrt{H_3}] \qquad (52)$$

将(52)式代入(49)式得到 $p(\boldsymbol{H})$。将 $p(\boldsymbol{H})$ 和 $T(\boldsymbol{H})$ 的值代入(36)式，得到系统(37)关于广义位移和广义动量的联合概率密度如下

$$p(\boldsymbol{q},\boldsymbol{p})=\frac{\omega_1\omega_2\omega_3\omega_4 p(\boldsymbol{H})}{8\pi^4}\sqrt[3]{\frac{k}{6H_3}}\Bigg|_{H_\eta=H_\eta(q_\eta,p_\eta),H_3=H_3(q_3,q_4,p_3,p_4)} \qquad (53)$$

其他统计量则可以由(53)式来导得，例如，第一个振子位移的稳态概率密度 $p(q_1)$，稳态均值 $E[Q_1]$ 及稳态均方值 $E[Q_1^2]$ 为

$$\begin{aligned}
p(q_1)&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}p(q,p)dp_1dp_2dp_3dp_4dq_2dq_3dq_4\\
E[Q_1]&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}q_1p(q,p)dp_1dp_2dp_3dp_4dq_1dq_2dq_3dq_4\\
E[Q_1^2]&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}q_1^2p(q,p)dp_1dp_2dp_3dp_4dq_1dq_2dq_3dq_4
\end{aligned} \qquad (54)$$

计算(54)式中的无穷积分时，还需要做一些积分变换。$p(q_i)$，$E[Q_i]$，$E[Q_i^2]$ $(i=2,3,4)$ 同理可得。令 $u_i^*=0$ 就可以得到系统(37)在未控情况下的稳态概率密度和统计量。

图 1-8 中给出了一些数值结果。各图中的理论计算结果(实线)与数字模拟结果(符号•o)都符合得很好。系统参数和激励强度为 $\alpha_{10}=\alpha_{20}=0.05$，$\alpha_{13}=\alpha_{14}=\alpha_{23}=\alpha_{24}=0.01$，$\alpha_{33}=\alpha_{34}=\alpha_{43}=\alpha_{44}=0$，$\alpha_{30}=\alpha_{40}=0.01$，$\alpha_{12}=\alpha_{21}=0.01$，$\alpha_{11}=\alpha_{22}=0$，$\alpha_{31}=\alpha_{32}=\alpha_{41}=\alpha_{42}=0.02$，$\omega_1^2=3$，$\omega_2^2=4$，$\omega_3^2=5$，$\omega_4^2=6$，$D_1=D_2=D_3=D_4=0.04$，$k=3$。从图中可以看出，控制力确实能降低系统(37)的响应。

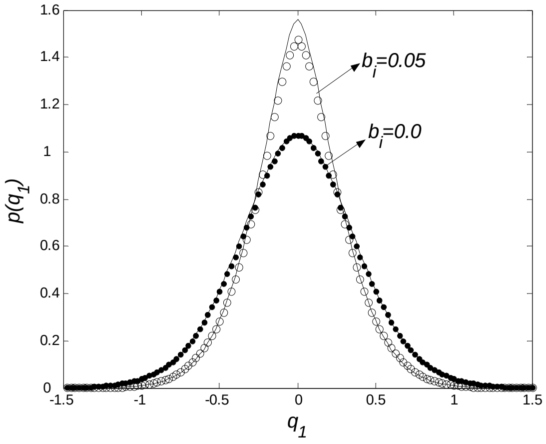

图 1 系统(37)第一个振子位移的稳态概率密度

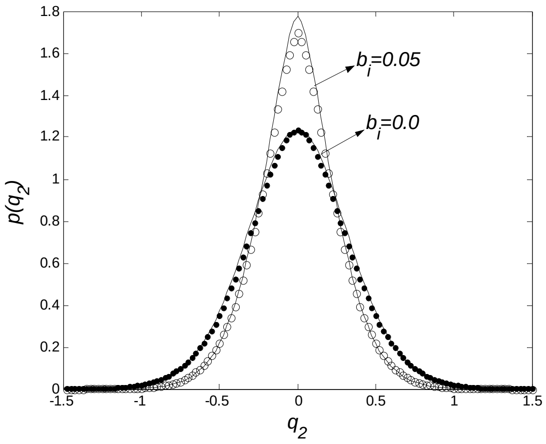

图 2 系统(37)第二个振子位移的稳态概率密度

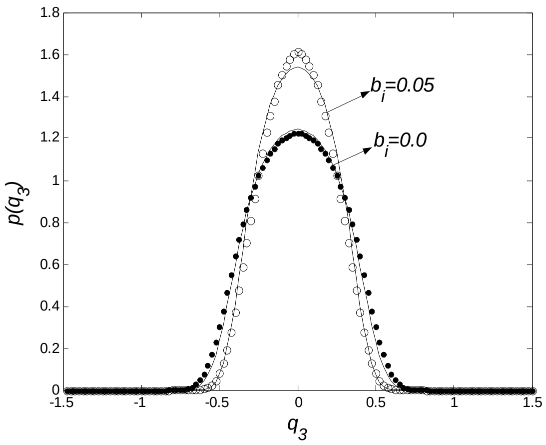

图 3 系统(37)第三个振子位移的稳态概率密度

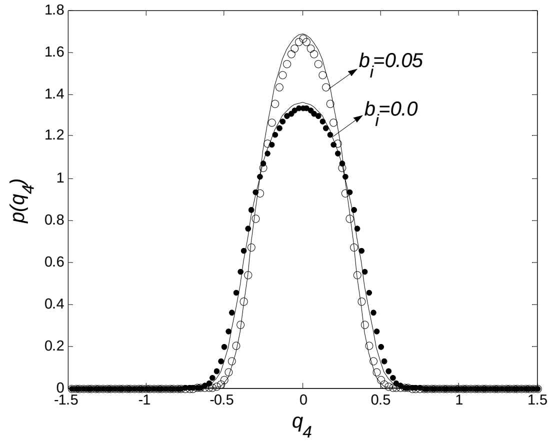

图 4 系统(37)第四个振子位移的稳态概率密度

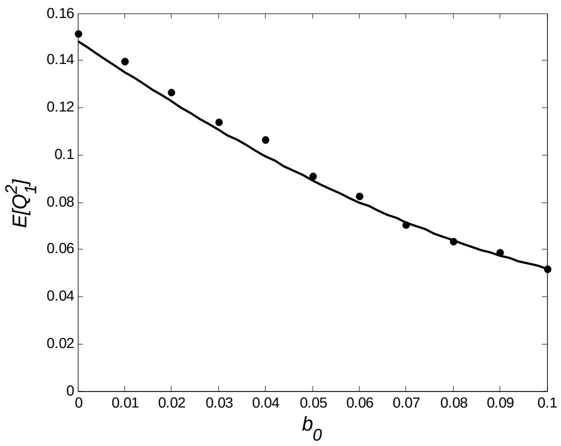

图 5 系统(37)第一个振子位移的稳态均方值随控制幅值的变化， ($b_i = b_0$)

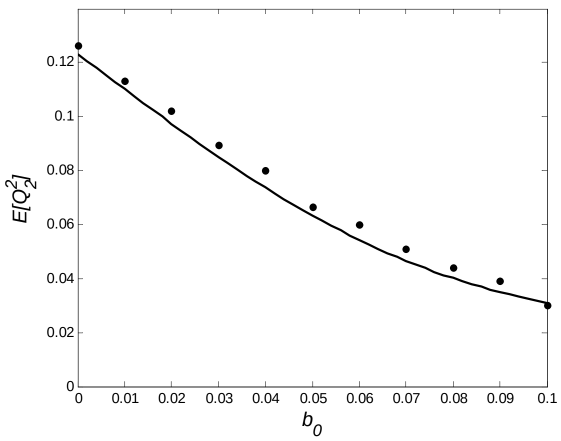

图 6 系统(37)第二个振子位移的稳态均方值随控制幅值的变化， ($b_i = b_0$)

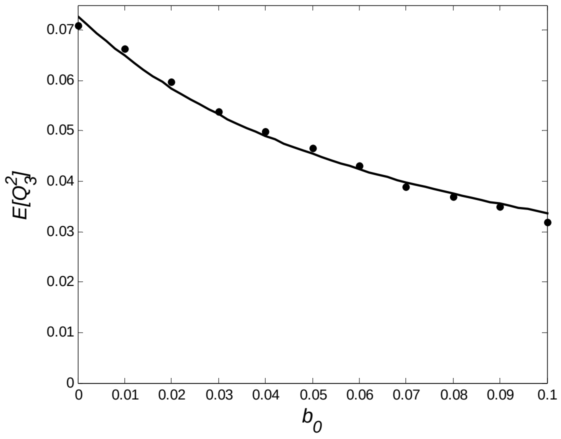

图 7 系统(37)第三个振子位移的稳态均方值随控制幅值的变化， $(b_i = b_0)$

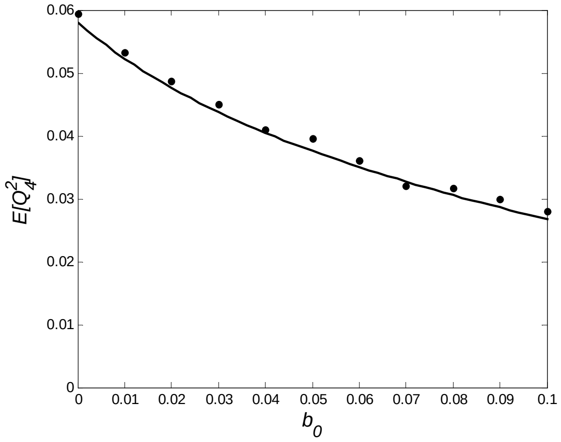

图 8 系统(37)第四个振子位移的稳态均方值随控制幅值的变化， $(b_i = b_0)$

6. 结论

本文提出了一种基于拟部分可积 Hamilton 系统的随机平均法和随机动态规划原理的最优有界控制策略。所提策略包括以下几个步骤：利用随机平均法将已知系统运动方程简化，降低方程的维数；在控制力为有界的条件下通过最小化动态规划方程得到最优控制规律；求解 FPK 方程得到受最优控制系统与未控系统的响应。在最后的算例里面，运用数字模拟结果充分验证了上面所提的步骤。尽管本文所研究的仅仅是拟部分可积 Hamilton 系统，但是这种最优控制策略也能运用于拟不可积 Hamilton 系统($r=1$)和拟可积 Hamilton 系统($r=n$)。因此，这种最优控制策略有很大的应用前景。

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Optimal bounded control of quasi partially-integrable Hamiltonian systems with stochastic excitations

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Abstract

A strategy for designing optimal bounded control to minimize the response of quasi partially-integrable Hamiltonian systems with stochastic excitations is proposed based on the stochastic averaging method for quasi partially-integrable Hamiltonian systems and the stochastic dynamical programming principle. First, the equations of motion of a controlled quasi partially- integrable Hamiltonian system with stochastic excitation are reduced to a set of partially- completed averaged Itô stochastic differential equations by using the stochastic averaging method for quasi partially-integrable Hamiltonian systems. Second, the dynamical programming equation is formulated by applying the stochastic dynamical programming principle. The optimal control law is derived from the dynamical programming equation and the control constraints without solving the dynamical programming equation. Finally, the response of optimally controlled system is predicted through solving the Fokker-Plank-Kolmogrov equation associated with fully-completed averaged Itô equations. An example is worked out in detail to illustrate the application and effectiveness of the proposed control strategy.

Keywords: *Nonlinear system, Stochastic excitation, Stochastic averaging, Stochastic optimal control, Dynamical programming*